

ROOT POLYTOPES AND ABELIAN IDEALS

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ABSTRACT. For any irreducible crystallographic root system Φ , we study the root polytope \mathcal{P}_Φ and the positive root polytope \mathcal{P}_Φ^+ , and show their relations with the abelian ideals of a Borel subalgebra of the complex simple Lie algebra with root system Φ , for types A and C . In particular, we give an algebraic construction of triangulations of \mathcal{P}_Φ and \mathcal{P}_Φ^+ whose simplices are distinguished abelian ideals. Moreover, for any irreducible root system Φ , we determine and study a hyperplane arrangement associated with Φ which is strictly related to \mathcal{P}_Φ .

1. INTRODUCTION

Let Φ be a finite irreducible crystallographic root system in a Euclidean space, Π a basis of Φ , and Φ^+ the corresponding set of positive roots. We denote by \mathcal{P}_Φ , or simply by \mathcal{P} , the convex hull of Φ and call \mathcal{P} the root polytope of Φ . Moreover, we denote by \mathcal{P}_Φ^+ , or simply by \mathcal{P}^+ , the convex hull of $\Phi^+ \cup \{\underline{0}\}$, where $\underline{0}$ is zero vector, and call \mathcal{P}^+ the positive root polytope of Φ . The root polytope, the positive root polytope, and their triangulations have been studied by several authors such as in [1], [15], [22], [23], for some or all the classical root systems. In [5], we have given a case free description of \mathcal{P} for arbitrary Φ . In this paper, we develop our algebraic-combinatorial analysis of \mathcal{P} , mostly for the types A_n and C_n , taking in account also the positive polytope. Our analysis relies on the results of our previous paper which we briefly recall.

Let \mathfrak{g} be a complex simple Lie algebra with Cartan subalgebra \mathfrak{h} and corresponding root system Φ , \mathfrak{g}_α the root space of α , for all $\alpha \in \Phi$, and \mathfrak{b} the Borel subalgebra of \mathfrak{g} corresponding to Φ^+ . Moreover, let W be the Weyl group of Φ . It is clear that W acts on the set of the faces of \mathcal{P} . In [5] we showed that there is a natural bijection between the set of the orbits of this action and a certain set of abelian ideals of \mathfrak{b} .

The abelian ideals of a Borel subalgebra of a complex simple Lie algebra are a widely studied subject. If \mathfrak{a} is a nontrivial abelian ideal of \mathfrak{b} , then there exists a subset $I_{\mathfrak{a}}$ of Φ^+ such that $\mathfrak{a} = \bigoplus_{\alpha \in I_{\mathfrak{a}}} \mathfrak{g}_\alpha$. If I is a subset of Φ^+ , then $\bigoplus_{\alpha \in I} \mathfrak{g}_\alpha$ is an abelian ideal of \mathfrak{b} if and only if I is a filter (or dual order ideal) of the poset

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Φ^+ with respect to the usual order of the root lattice, and moreover, the sum of any two roots in I is not a root. We call a subset of Φ^+ with these properties, by abuse, an abelian ideal of Φ^+ .

We proved that each orbit of the action of W on the set of the faces of \mathcal{P} contains a distinguished face, which is the convex hull of an abelian ideal of Φ^+ , and we determined explicitly the abelian ideals corresponding to these faces of \mathcal{P} . Moreover, we proved that these abelian ideals are all principal i.e., of type $M_\alpha := \{\beta \in \Phi^+ \mid \beta \geq \alpha\}$, for some $\alpha \in \Phi^+$. We call the faces corresponding to these abelian ideals the standard parabolic faces. Thus, the standard parabolic faces are a set of representatives of the orbits of the faces of \mathcal{P} under the action of W .

The root polytopes of both the types A_n and C_n have similar special properties, with respect to the others. As usual, we call facets the maximal proper faces of \mathcal{P} , i.e. the faces of codimension 1. The first special property is that in these cases all the abelian ideals corresponding to facets of \mathcal{P} are maximal, and are all the maximal abelian ideals in Φ^+ . This also means, in these two cases, that the standard parabolic facets of \mathcal{P} are exactly the convex hulls of the abelian ideals M_α , for all the simple long roots α .

The poset of the abelian ideals with respect to the inclusion is quite well known (see [24], [27] and [9] for results and motivations). In the cases A_n and C_n , we find a natural bijection between a special subposet of the abelian ideals and a certain triangulation of the standard parabolic facets of \mathcal{P} . If F is any standard parabolic facet and W_F is its stabilizer in W , then we can extend the triangulation of F to its orbit by means of any system of representatives of the set of left cosets W/W_F . If we choose, for all the facets, the system of minimal length representatives, we obtain a triangulation of the whole \mathcal{P} which is the easiest one from a combinatorial point of view. This triangulation is already studied with a combinatorial approach in [1]. Our construction provides also a simple algebraic proof that this triangulation restricts to a triangulation of the positive root polytope \mathcal{P}^+ . This proves in particular that, for Φ of type A_n and C_n , \mathcal{P}^+ is the intersection of \mathcal{P} with the positive cone generated by the positive roots, which is false in general. For type A_n , the triangulation obtained for \mathcal{P}^+ is the one corresponding to the antistandard bases of [15].

The subposets of the abelian ideals corresponding to the triangulations of the standard parabolic facets have an important role in the study of the maximal abelian ideals. We need some more preliminaries for describing it.

We recall the following correspondence, which has been established by Peterson but described in [15]. To any abelian ideal \mathfrak{i} of \mathfrak{b} we can associate an element $w_{\mathfrak{i}}$ of the affine Weyl group \widehat{W} of Φ , so obtaining a one to one correspondence

between the set \mathcal{I}_{ab} of the abelian ideals of \mathfrak{b} and a subset \widehat{W}_{ab} of \widehat{W} . For the affine root system associated to Φ , we adopt the notation and definitions of [20, Chapter 6], if not otherwise specified. In particular, we denote by δ the unitary imaginary root and by θ the highest root of Φ , and set $\alpha_0 = \delta - \theta$, so that $\widehat{\Pi} := \{\alpha_0\} \cup \Pi$ is a simple system for the (untwisted) affine root system associated to Φ . In the study of the poset of the abelian ideals, a special role is played by the sets $\mathcal{I}_\alpha := \{\mathfrak{i} \in \mathcal{I}_{ab} \mid w_{\mathfrak{i}}^{-1}(2\delta - \theta) = \alpha\}$, for all long α in Π . Each such \mathcal{I}_α has a maximum, and the maximal elements of the \mathcal{I}_α , for all long α in Π , are exactly the maximal abelian ideals. Moreover, the poset structure of the \mathcal{I}_α is well known. To each abelian ideal \mathfrak{i} in \mathcal{I}_α we associate a simplex $\sigma_{\mathfrak{i}}$, in such a way that the resulting set of simplices yields a triangulation of the standard parabolic facet corresponding to M_α . This triangulation inherits, in a certain sense, the poset structure of \mathcal{I}_α . In fact, if $\mathfrak{j} \in \mathcal{I}_\alpha$, then the set of all simplices $\sigma_{\mathfrak{i}}$ with $\mathfrak{i} \subseteq \mathfrak{j}$ yields a triangulation of the convex hull of \mathfrak{j} .

From our construction, we also obtain simple proofs of known and less known results about the volumes of \mathcal{P} and \mathcal{P}^+ .

A second special property of the types A_n and C_n is related to a certain hyperplane arrangement associated to \mathcal{P} . For any root type, using the explicit description of the faces of \mathcal{P} given in [5], we compute the set of all hyperplanes through the origin that contain some face of codimension 2. We denote by \mathcal{H}_Φ , or simply by \mathcal{H} , the resulting arrangement. By construction, for each facet F of \mathcal{P} , the halfcone centered at the origin generated by F is the closure of a union of regions of \mathcal{H} . For the types A_n and C_n , we show that the cone generated by each facet is the closure of a single region of \mathcal{H} , so that there is a natural bijection between the regions of \mathcal{H} and the facets of \mathcal{P} . The analogous result does not hold for the type B_n and D_n . Moreover, in both the cases A_n and C_n , \mathcal{H} is the orbit of ω_1^\perp under the action of W , where ω_1 is the first fundamental weight.

The paper is organized as follows. In Section 2, we fix the notation and we recall the results that we most frequently use in the paper. In Section 3, we introduce the hyperplane arrangement \mathcal{H}_Φ and we explain how to explicitly construct it for all irreducible root systems Φ . In Section 4, we analyze the relation between the root polytope \mathcal{P}_Φ and the arrangement \mathcal{H}_Φ : in particular, we show that the cones on the facets of \mathcal{H}_Φ are precisely the closures of the regions of \mathcal{H}_Φ , for types A and C , while they are union of closures of regions, for types B and D . In Section 5, we study the principal maximal abelian ideal of the Borel subalgebra of an arbitrary complex simple Lie algebra \mathfrak{g} . Then we explain how the maximal abelian ideals of the Borel subalgebra are related to the associated root polytope \mathcal{P} , for types A and C . The proofs are given in Section 6 for type A and in Section 8 for type C . In Section 7, we show how the simplices of the triangulation studied

in Section 6 can be interpreted as directed graphs. In Section 9, we show how the curious identity of [11] holds also for the triangulations studied in Sections 6 and 8.

2. PRELIMINARIES

In this section, we fix the notation and recall the known results that we most frequently use in the paper.

For $n, m \in \mathbb{Z}$, with $n \leq m$, we set $[n, m] = \{n, n+1, \dots, m\}$ and, for $n \in \mathbb{N} \setminus \{0\}$, $[n] = [1, n]$. For every set I , we denote its cardinality by $|I|$.

For basic facts about root systems, Weyl groups, Lie algebras, and convex polytopes, we refer the reader, respectively, to [4], [3] and [18], [17], and [16].

2.1. Root systems. Let Φ be a finite irreducible (reduced) crystallographic root system in a Euclidean space E with the positive definite bilinear form $(-, -)$.

For all $X \subseteq E$, we denote by $\text{Span } X$ the real vector space generated by X . Thus $E = \text{Span } \Phi$.

We fix our further notation on the root system and its Weyl group in the following list:

n	the rank of Φ ,
$\Pi = \{\alpha_1, \dots, \alpha_n\}$	set of simple roots,
Π_ℓ	the set of long simple roots,
$\Omega = \{\check{\omega}_1, \dots, \check{\omega}_n\}$	the set of fundamental coweights (the dual basis of Π),
Φ^+	the set of positive roots w.r.t. Π ,
$\Phi(\Gamma)$	the root subsystem generated by Γ , for all $\Gamma \subseteq \Phi$,
$\Phi^+(\Gamma)$	$= \Phi(\Gamma) \cap \Phi^+$,
$\overline{\Phi'}$	$= \text{Span } \Phi' \cap \Phi$, the parabolic closure of the subsystem Φ' ,
$c_\alpha(\beta), c_i(\beta)$	the coordinates of $\beta \in \Phi$ w.r.t. $\Pi : \beta = \sum_{\alpha \in \Pi} c_\alpha(\beta)\alpha$, $c_i(\beta) = c_{\alpha_i}(\beta)$,
$\text{Supp}(\beta)$	$= \{\alpha \in \Pi \mid c_\alpha(\beta) \neq 0\}$,
$ht(\alpha)$	$= \sum_{i=1}^n c_i(\alpha)$, the height of the root α ,
θ	the highest root,
m_α, m_i	$= c_\alpha(\theta)$, $m_i = m_{\alpha_i}$,
W	the Weyl group of Φ ,
s_α	the reflection through the hyperplane α^\perp ,
ℓ	the length function of W w.r.t. $\{s_\alpha \mid \alpha \in \Pi\}$,
w_0	the longest element of W ,
$D_r(w)$	$= \{\alpha \in \Pi \mid \ell(ws_\alpha) < \ell(w)\}$, the set of right descents of w ,
$N(w)$	$= \{\beta \in \Phi^+ \mid w^{-1}(\beta) \in -\Phi^+\}$
$\overline{N}(w)$	$= \{\beta \in \Phi^+ \mid w(\beta) \in -\Phi^+\}$,

$W\langle\Gamma\rangle$	the subgroup of W generated by $\{s_\alpha \mid \alpha \in \Gamma\}$ ($\Gamma \subseteq \Phi$),
$\widehat{\Phi}$	the affine root system associated with Φ ,
α_0	the affine simple root of $\widehat{\Phi}$,
$\widehat{\Pi}$	$= \Pi \cup \{\alpha_0\}$,
$\widehat{\Phi}^+$	the set of positive roots of $\widehat{\Phi}$ w.r.t. $\widehat{\Pi}$,
\widehat{W}	the affine Weyl group of Φ .

We call *integral basis* a basis of the vector space $\text{Span } \Phi$ that also is a \mathbb{Z} -basis of the root lattice $\sum_{i \in [n]} \mathbb{Z}\alpha_i$.

We denote by \leq the usual order of the root lattice: $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a nonnegative linear combination of roots in Φ^+ .

We call a set N of positive roots *convex* if all roots that are a positive linear combination of roots in N belong to N . It is clear that, for all w in the Weyl group W , the set

$$N(w) = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) < 0\}$$

and its complement $\Phi^+ \setminus N(w)$ are both convex.

A set N of positive roots is called *closed* if all roots that are a sum of two roots in N belong to N . It is well known that if N and $\Phi^+ \setminus N$ are both closed, then there exists $w \in W$ such that $N = N(w)$.

Since a convex set is closed, for all $N \subseteq \Phi^+$ the following three conditions are equivalent:

- (1) N and $\Phi^+ \setminus N$ are closed,
- (2) there exists $w \in W$ such that $N = N(w)$,
- (3) N and $\Phi^+ \setminus N$ are convex.

For all $w \in W$, the convexity of $\Phi^+ \setminus N(w)$ implies, in particular, that for all $\beta \in N(w)$

$$(2.1) \quad \text{Supp}(\beta) \cap N(w) \neq \emptyset.$$

We recall that, if $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$ is a reduced expression of $w \in W$, then

$$(2.2) \quad \begin{aligned} N(w) &= \{\gamma_1, \dots, \gamma_r\}, \quad \text{where} \\ \gamma_1 &= \alpha_{i_1}, \quad \gamma_h = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_{h-1}}}(\alpha_{i_h}) \text{ for } h \in [2, r]. \end{aligned}$$

By definition, $\overline{N}(w) = N(w^{-1})$, hence $\overline{N}(w) = \{\beta_1, \dots, \beta_r\}$, where $\beta_r = \alpha_{i_r}$, $\beta_h = s_{\alpha_{i_r}} \cdots s_{\alpha_{i_{h+1}}}(\alpha_{i_h})$, for $h \in [r-1]$. We also recall that right descents of $w \in W$ are the simple roots in $\overline{N}(w)$:

$$(2.3) \quad D_r(w) = \overline{N}(w) \cap \Pi.$$

2.2. Ideals. By the root poset of Φ (w.r.t. the basis Π) we intend the partially ordered set whose underlying set is Φ^+ , with the order induced by the root lattice. The root poset could be equivalently defined as the transitive closure of the relation $\alpha \triangleleft \beta$ if and only if $\beta - \alpha$ is a simple root. The root poset hence is ranked by the height function and has the highest root θ as maximum.

A *dual order ideal*, or *filter*, of Φ^+ is a subset I of Φ^+ such that, if $\alpha \in I$ and $\beta \geq \alpha$, then $\beta \in I$.

Let \mathfrak{g} be a complex simple Lie algebra, and \mathfrak{h} a Cartan subalgebra of \mathfrak{g} such that Φ is the root system of \mathfrak{g} with respect to \mathfrak{h} . For all $\alpha \in \Phi$, we denote by \mathfrak{g}_α the root space of α . Moreover, we denote by \mathfrak{b} the Borel subalgebra corresponding to Φ^+ : $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$, where $\mathfrak{n}^+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha = [\mathfrak{b}, \mathfrak{b}]$.

Let \mathfrak{i} be an ideal of \mathfrak{b} . Then \mathfrak{i} is \mathfrak{h} -stable, hence there exists a subset $\Phi_{\mathfrak{i}}$ of Φ^+ such that

$$\mathfrak{i} = \sum_{\alpha \in \Phi_{\mathfrak{i}}} \mathfrak{g}_\alpha + \mathfrak{i} \cap \mathfrak{h}.$$

The condition $[\mathfrak{i}, \mathfrak{b}] \subseteq \mathfrak{i}$ implies that $(\Phi_{\mathfrak{i}} + \Phi^+) \cap \Phi \subseteq \Phi_{\mathfrak{i}}$, i.e. that $\Phi_{\mathfrak{i}}$ is a dual order ideal in the root poset. Conversely, if I is a dual order ideal of Φ^+ , then

$$\mathfrak{i}_I = \sum_{\alpha \in I} \mathfrak{g}_\alpha$$

is an ideal of \mathfrak{b} , included in \mathfrak{n}^+ . We call *ad-nilpotent* the ideals of \mathfrak{b} included in \mathfrak{n}^+ . Thus $\mathfrak{i} \mapsto \Phi_{\mathfrak{i}}$ is a bijection from the set of ad-nilpotent ideals of \mathfrak{b} to the set of dual order ideals of Φ^+ , with inverse map $I \mapsto \mathfrak{i}_I$.

It is easy to prove an abelian ideal of \mathfrak{b} is necessarily *ad-nilpotent*. The ideal \mathfrak{i} is abelian if and only $\Phi_{\mathfrak{i}}$ satisfies the abelian condition: $(\Phi_{\mathfrak{i}} + \Phi_{\mathfrak{i}}) \cap \Phi = \emptyset$.

The dual order ideal I of Φ^+ is called *principal* if it has a minimum. In such a case the ad-nilpotent ideal \mathfrak{i}_I is principal, being generated, as a \mathfrak{b} -module, by any non-zero vector of the root space \mathfrak{g}_η , where $\eta = \min I$.

Following [20], we denote by δ the basic imaginary root, so that $\text{Span } \widehat{\Phi} = \text{Span } \Phi \oplus \mathbb{R}\delta$, and denote by $\widehat{\Phi}$ the set of real roots of the untwisted affine root system associated with Φ ,

$$\widehat{\Phi} = \Phi + \mathbb{Z}\delta := \{\alpha + k\delta \mid \alpha \in \Phi, k \in \mathbb{Z}\}.$$

Then $\widehat{\Phi}$ is a crystallographic affine root system in $\text{Span } \widehat{\Phi}$ endowed with the bilinear form obtained by extending the scalar product of $\text{Span } \Phi$ to a positive semidefinite form with kernel $\mathbb{R}\delta$. If we take $\alpha_0 = -\theta + \delta$, then $\widehat{\Pi} := \{\alpha_0\} \cup \Pi$ is a root basis for $\widehat{\Phi}$. The set of positive roots of $\widehat{\Phi}$ with respect to $\widehat{\Pi}$ is $\widehat{\Phi}^+ := \Phi^+ \cup (\Phi + \mathbb{Z}^+\delta)$, where \mathbb{Z}^+ is the set of positive integers. The affine Weyl group \widehat{W} associated to W is the Weyl group of $\widehat{\Phi}$. We extend the notation $N(w)$ to all $w \in \widehat{W}$, so $N(w) = \{\alpha \in \widehat{\Phi}^+ \mid w^{-1}(\alpha) \in -\widehat{\Phi}^+\}$.

We recall Peterson's construction described in [21]. Let \mathfrak{i} be an abelian ideal of \mathfrak{b} , and $-\Phi_{\mathfrak{i}} + \delta := \{-\alpha + \delta \mid \alpha \in \Phi_{\mathfrak{i}}\}$. There exists a (unique) element $w_{\mathfrak{i}} \in \widehat{W}$ such that $N(w_{\mathfrak{i}}) = -\Phi_{\mathfrak{i}} + \delta$. Moreover, the map $\mathfrak{i} \mapsto w_{\mathfrak{i}}$ is a bijection from the set of the abelian ideals of \mathfrak{b} and the set of all elements w in \widehat{W} such that $N(w) \subseteq -\Phi^+ + \delta$.

In order to simplify the notation, we identify the ad-nilpotent ideals with their set of roots: henceforward, we shall view such ideals as subsets of Φ^+ .

We denote by \mathcal{I}_{ab} the set of abelian ideals in Φ^+ and by \widehat{W}_{ab} the corresponding set of elements in \widehat{W} .

For all $w \in \widehat{W}_{ab}$, $w^{-1}(-\theta + 2\delta)$ is a long positive root and belongs to Φ . For all long roots $\alpha \in \Phi^+$, let $\mathcal{I}_{ab}(\alpha)$ be the subset of all $\mathfrak{i} \in \mathcal{I}_{ab}$ such that $w_{\mathfrak{i}}^{-1}(-\theta + 2\delta) = \alpha$.

In the following statement we collect some results about the poset of the abelian ideals proved by Panyushev.

Theorem 2.1. (*Panyushev [24]*).

- (1) *For any long root α , $\mathcal{I}_{ab}(\alpha)$ has a minimum and a maximum.*
- (2) *The maximal abelian ideals are exactly the maximal elements of the $\mathcal{I}_{ab}(\alpha)$ with $\alpha \in \Pi_{\ell}$. In particular, the maximal abelian ideals are in bijection with the long simple roots.*
- (3) *For any pair of distinct α, α' in Π_{ℓ} , any ideal in $\mathcal{I}_{ab}(\alpha)$ is incomparable with any ideal in $\mathcal{I}_{ab}(\alpha')$.*

The following easy result will be needed in the next sections.

Proposition 2.2. *Let $\mathfrak{i} \in \mathcal{I}_{ab}$. Then there exists $\alpha \in \Pi_{\ell}$ such that $\mathfrak{i} \in \mathcal{I}_{ab}(\alpha)$ if and only if, for all β, γ in Φ^+ such that $\beta + \gamma = \theta$, exactly one of β and γ belongs to \mathfrak{i} .*

Proof. Assume that for all β, γ in Φ^+ such that $\beta + \gamma = \theta$, exactly one of β and γ belongs to \mathfrak{i} . Notice that $\beta', \gamma' \in \widehat{\Phi}^+$ are such that $\beta' + \gamma' = -\theta + 2\delta$ if and only if there exist $\beta, \gamma \in \Phi^+$ such that $\beta' = \delta - \beta$, $\gamma' = \delta - \gamma$ and $\beta + \gamma = \theta$. Therefore, the assumption is equivalent to the fact that for all $\beta', \gamma' \in \widehat{\Phi}^+$ such that $\beta' + \gamma' = -\theta + 2\delta$, exactly one of β', γ' belong to $N(w_{\mathfrak{i}})$. Now, it is easily seen that this condition is equivalent to the fact that $N(w_{\mathfrak{i}}) \cup \{-\theta + 2\delta\}$ is a biconvex set, hence that there exists $w \in \widehat{W}$ such that $N(w_{\mathfrak{i}}) \cup \{-\theta + 2\delta\} = N(w)$. By equations 2.2, this happens if and only if there exists a simple root α such that $w = w_{\mathfrak{i}}s_{\alpha}$ and $w_{\mathfrak{i}}(\alpha) = -\theta + 2\delta$. \square

2.3. Root polytopes. For any subset S of $\text{Span}(\Phi)$, we denote by $\text{Conv}(S)$ the convex hull of S and by $\text{Conv}_0(S)$ the convex hull of $S \cup \{0\}$.

We denote by \mathcal{P}_Φ the root polytope of Φ , i.e. the convex hull of all roots in Φ :

$$\mathcal{P}_\Phi := \text{Conv}(\Phi).$$

We recall the results in [5] that will be needed in the sequel. Recall that $\theta = \sum_{i \in [n]} m_i \alpha_i$ is the highest root of Φ , with respect to Π , and $\{\check{\omega}_i \mid i \in [n]\}$ is the set of the fundamental co-weights of Φ . We will think of $\check{\omega}_i$ both as a functional and as a vector (the vector of $\text{Span } \Phi$ defined by the condition $(\alpha_j, \check{\omega}_i) = \delta_{ji}$, for all $j \in [n]$).

For every $I \subseteq [n]$, we let

$$V_I := \{\alpha \in \Phi^+ \mid (\theta, \check{\omega}_i) = m_i, \forall i \in I\}, \quad F_I := \text{Conv}(V_I)$$

and we call F_I the *standard parabolic face associated with I* . For simplicity, we let $F_i = F_{\{i\}}$. The standard parabolic faces F_I are actually faces of the root polytope \mathcal{P}_Φ , since \mathcal{P}_Φ is included in the half-space $(-, \check{\omega}_i) \leq m_i$. Moreover, the maximal root θ belongs to all standard parabolic faces.

We denote by $\widehat{\Phi}$ the affine root system associated with Φ . Let $\widehat{\Pi}$ be an extension of Π to a simple system of $\widehat{\Phi}$ and $\alpha_0 \in \widehat{\Phi}$ be such that $\widehat{\Pi} = \Pi \cup \{\alpha_0\}$. Given $I \subseteq [n]$, we let $\Pi_I := \{\alpha_i \mid i \in I\}$, $\Gamma_0(I)$ be the connected component of α_0 in $\widehat{\Pi} \setminus \Pi_I$, and

- $\bar{I} := \{k \mid \alpha_k \notin \Gamma_0(I)\}$,
- $\partial I := \{j \mid \alpha_j \in \Pi_I, \text{ and } \exists \beta \in \Gamma_0(I) \text{ such that } \beta \not\leq \alpha_j\}$.

We recall that, for any subset Γ of Φ , we denote by $W\langle\Gamma\rangle$ the subgroup of W generated by the reflections with respect to the roots in Γ , and by $\Phi(\Gamma)$ the root subsystem generated by Γ , i.e. $\Phi(\Gamma) := \{w(\gamma) \mid w \in W\langle\Gamma\rangle, \gamma \in \Gamma\}$. For each $\Gamma \subseteq \Pi$, let $\widehat{\Gamma} = \Gamma \cup \{\alpha_0\}$ and $\widehat{\Phi}(\widehat{\Gamma})$ be the standard parabolic subsystem of $\widehat{\Phi}$ generated by $\widehat{\Gamma}$.

In the following result we collect several properties of the standard parabolic faces (see [5] for proofs).

Theorem 2.3. *Let $I \subset [n]$. Then:*

- (1) V_I is a principal abelian ideal in Φ^+ ,
- (2) $\{J \subset [n] \mid F_J = F_I\} = \{J \subset [n] \mid \partial I \leq J \leq \bar{I}\}$,
- (3) the dimension of F_I is $n - |\bar{I}|$,
- (4) the stabilizer of F_I in W is $W\langle\Pi \setminus \Pi_{\partial I}\rangle$,
- (5) $\{\bar{I} \mid I \subseteq [n]\} = \{I \subseteq [n] \mid \widehat{\Phi}(\widehat{\Pi} \setminus \Pi_I) \text{ is irreducible}\}$,
- (6) the set $\{\Pi \setminus \Pi_{\bar{I}}\} \cup \{-\theta\}$ is a basis of the root subsystem $\Phi(V_I)$ generated by the roots in F_I .

Moreover, the faces F_I , for $I \in \mathcal{F} = \{I \subseteq [n] \mid \widehat{\Phi}(\widehat{\Pi} \setminus \Pi_I) \text{ is irreducible}\}$, form a complete set of representatives of the W -orbits. In particular, the f -polynomial

of \mathcal{P}_Φ is

$$\sum_{I \in \mathcal{F}} [W : W \langle \Pi \setminus \Pi_{\partial I} \rangle] t^{n-|I|}.$$

The explicit description of the facets yields a description of \mathcal{P}_Φ as an intersection of a minimal set of half-spaces.

Corollary 2.4. *Let $\Pi_{\mathcal{F}} = \left\{ \alpha \in \Pi \mid \widehat{\Phi} \left(\widehat{\Pi} \setminus \{\alpha\} \right) \text{ is irreducible} \right\}$ and let $\mathcal{L}(W^\alpha)$ be a set of representatives of the left cosets of W modulo the subgroup $W \langle \Pi \setminus \{\alpha\} \rangle$. Then*

$$\mathcal{P}_\Phi = \{x \mid (x, w\check{\omega}_\alpha) \leq m_\alpha, \text{ for all } \alpha \in \Pi_{\mathcal{F}} \text{ and } w \in \mathcal{L}(W^\alpha)\}.$$

Moreover, the above one is the minimal set of linear inequalities that defines \mathcal{P}_Φ as an intersection of half-spaces.

2.4. Hyperplane arrangements. We follow [26] for notation and terminology concerning hyperplane arrangements. Given an hyperplane arrangement \mathcal{H} in \mathbb{R}^n , a region of \mathcal{H} is a connected component of the complement $\mathbb{R}^n \setminus \cup_{H \in \mathcal{H}} H$ of the hyperplanes. We let $\mathcal{L}(\mathcal{H})$ be the intersection poset of \mathcal{H} , i.e. the set of all nonempty intersection of hyperplanes in \mathcal{H} (including \mathbb{R}^n as the intersection over the empty set) partially ordered by reverse inclusion and $\chi_{\mathcal{H}}(t)$ be the characteristic polynomial of \mathcal{H} :

$$\chi_{\mathcal{H}}(t) = \sum_{x \in \mathcal{L}(\mathcal{H})} \mu(x) t^{\dim x}$$

($\mu(x)$ is the Möbius function from the bottom element \mathbb{R}^n of $\mathcal{L}(\mathcal{H})$). A face of \mathcal{H} is a set $\emptyset \neq F = \overline{R} \cap x$, where \overline{R} is the closure of a region R and $x \in \mathcal{L}(\mathcal{H})$. We denote by $f_{\mathcal{H}}(x)$ the face polynomial of \mathcal{H} : $f_{\mathcal{H}}(x) = \sum x^{\dim F}$ (the sum is over all faces F).

The following result is due to Zaslavsky [29].

Theorem 2.5. *The number of regions of an arrangement \mathcal{H} in an n -dimensional real vector space is $(-1)^n \chi_{\mathcal{H}}(-1)$.*

3. A HYPERPLANE ARRANGEMENT ASSOCIATED TO \mathcal{P}_Φ

For all irreducible root systems Φ , we consider a central hyperplane arrangement \mathcal{H}_Φ associated with the root polytope \mathcal{P}_Φ and hence with Φ . The hyperplanes of \mathcal{H}_Φ are the hyperplanes through the origin spanned by the faces of \mathcal{P}_Φ of codimension 2:

$$\mathcal{H}_\Phi := \{ \text{Span } F \mid F \text{ face of } \mathcal{P}_\Phi \text{ and } \text{codim } F = 2 \}.$$

For any face F , we denote by V_F the roots lying on F :

$$V_F = F \cap \Phi.$$

The long roots in V_F are the vertices of the face F , hence $\text{Span } F = \text{Span } V_F$. Clearly, the following result holds.

Proposition 3.1. *Let Φ be an irreducible root system. Each cone on a facet of \mathcal{P}_Φ is the closure of a union of regions of the hyperplane arrangement \mathcal{H}_Φ .*

We are going to compute explicitly \mathcal{H}_Φ for all irreducible Φ . It is clear that \mathcal{H}_Φ is W -stable. In fact, by Theorem 2.3, \mathcal{H}_Φ consists of the orbits of the hyperplanes spanned by the standard parabolic faces of codimension 2 for the action of W . By Theorem 2.3, (3) and (5), such faces are exactly the F_I with $I = \{i, j\}$ and $\widehat{\Pi} \setminus \{\alpha_i, \alpha_j\}$ irreducible. In Table 1, we list all the irreducible $(n-1)$ -dimensional root subsystems obtained by removing two finite nodes from the extended Dynkin diagram. If α_i and α_j correspond to the crossed nodes and we read the affine node as $-\theta$, then the resulting subsystem is the $(n-1)$ -dimensional subsystem of Φ with root basis $\{\alpha_k \mid k \neq i, j\} \cup \{-\theta\}$. Thus, these are exactly the root subsystems $\Phi(V_F)$, i.e. the subsystems generated by the V_F , for all the standard parabolic $(n-2)$ -faces F .

Proposition 3.2. *Let Φ be an irreducible root system. There exists a subset $H_\Phi \subseteq [n]$ such that*

$$\mathcal{H}_\Phi = \{w(\check{\omega}_k)^\perp \mid w \in W, k \in H_\Phi\}.$$

Furthermore, H_Φ is contained in the set of $h \in [n]$ such that the standard parabolic subsystem of Φ generated by $\Pi \setminus \{\alpha_h\}$ is irreducible.

Proof. Let F be a $(n-2)$ -face of \mathcal{P} . By Theorem 2.3, the roots in F generate an irreducible subsystem $\Phi(V_F)$ of rank $n-1$. Hence also the parabolic closure $\overline{\Phi(V_F)} = \text{Span } F \cap \Phi$ is irreducible and has rank $n-1$. It follows that there exist a $k \in [n]$ and a w in W such that the standard parabolic subsystem generated by $\Pi \setminus \{\alpha_k\}$ is irreducible and is transformed by w into $\overline{\Phi(V_F)}$. Hence $\text{Span } F = w(\check{\omega}_k)^\perp$. \square

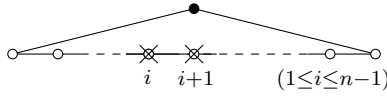
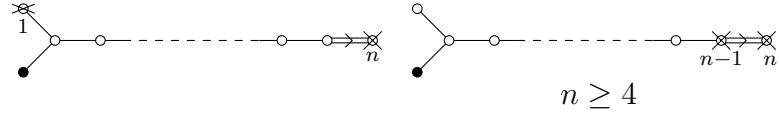
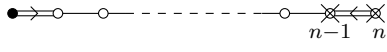
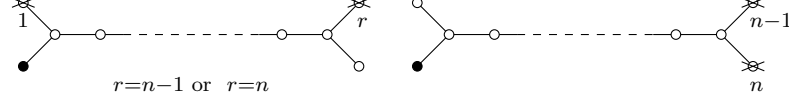
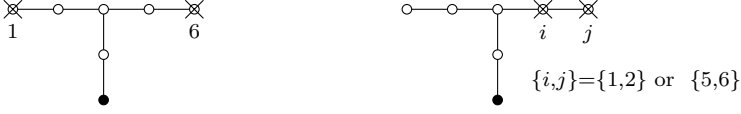

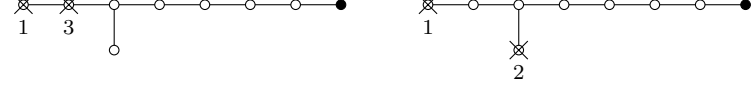
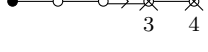
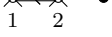
We remark that though different standard parabolic faces F and F' belong to different orbits under the action of W (Theorem 2.3), it may happen that the hyperplanes $\text{Span } F$ and $\text{Span } F'$ belong to the same orbit of W . This happens in type A_n and D_n .

For $k \in [n]$, let $[\check{\omega}_k^\perp]$ be the orbit of $\check{\omega}_k^\perp$ under the action of W ,

$$[\check{\omega}_k^\perp] := \{w(\check{\omega}_k)^\perp \mid w \in W\}.$$

We notice that, for all $k, j \in [n]$, $[\check{\omega}_k^\perp] = [\check{\omega}_j^\perp]$ if and only if the standard parabolic subsystems $\Phi(\Pi \setminus \{\alpha_k\})$ and $\Phi(\Pi \setminus \{\alpha_j\})$ can be transformed each into the other by W . In fact, $\Phi(\Pi \setminus \{\alpha_k\}) = \check{\omega}_k^\perp \cap \Phi$ and hence for each $w \in W$,

TABLE 1. Subdiagrams corresponding to the codimension 2 faces

A_n	
B_n	
C_n	
D_n	
E_6	
E_7	
E_8	
F_4	
G_2	

$w(\check{\omega}_k^\perp) = \check{\omega}_j^\perp$ if and only if $w(\Phi(\Pi \setminus \{\alpha_k\})) = \Phi(\Pi \setminus \{\alpha_j\})$. In particular, if $\Phi(\Pi \setminus \{\alpha_k\})$ and $\Phi(\Pi \setminus \{\alpha_j\})$ have different Dynkin diagrams, then $[\check{\omega}_k^\perp] \neq [\check{\omega}_j^\perp]$.

However, in general the Dynkin graph does not determine the orbit of the parabolic subsystem; standard parabolic subsystems having isomorphic Dynkin diagrams may be or not be transformed one into the other by the Weyl group. Thus, by Proposition 3.2, in order to determine \mathcal{H}_Φ we need to determine, for each standard parabolic $(n-2)$ -face F , the parabolic closure $\overline{\Phi(V_F)}$ and a standard

parabolic subsystem in its orbit under the action of W . Moreover, in order to determine a set of representatives of the orbits of \mathcal{H}_Φ , we must check whether these standard parabolic can be transformed each into the other by W .

For computing explicitly the parabolic closures of the subsystems in Table 1, we use the following lemma.

Lemma 3.3. *Let $I \subseteq [n]$ be such that $\dim F_I = n - |I|$. For any $b \in \mathbb{R}^+$, let $S_b = \{\gamma \in \Phi \mid c_i(\gamma) = \frac{m_i}{b} \ \forall i \in I\}$, and let d be the maximal positive number such that $S_d \neq \emptyset$.*

Then:

- (1) S_d has a minimum (in the root poset) α and $\{\alpha\} \cup \Pi'_I$ is a root basis for $\overline{\Phi(V_I)}$;
- (2) $\Phi(V_I) = \overline{\Phi(V_I)}$ if and only if $d = 1$. In particular, if $\gcd\{m_i \mid i \in I\} = 1$, then $\Phi(V_I) = \overline{\Phi(V_I)}$.

Proof. Let $b \in \mathbb{R}^+$ be such that $S_b \neq \emptyset$. First notice that $\frac{m_i}{b}$ must be an integer between 1 and m_i , for all $i \in I$, and, in particular, b must be a rational number between 1 and $\min\{m_i \mid i \in I\}$. Moreover, if $b = z/t$ with z and t relatively prime integers, then $z \mid m_i$ for all $i \in I$: therefore, if $\gcd\{m_i \mid i \in I\} = 1$, then $d = 1$. Recall from Theorem 2.3 that $\Phi(V_I)$ has basis $\{\Pi \setminus \Pi_I\} \cup \{-\theta\}$, hence $\Phi(V_I) = S_1^\pm \cup \Phi(\Pi \setminus \Pi_I)$, where we set $S_b^\pm = S_b \cup -S_b$, for all b . Moreover, since $\text{Span } V_I$ is generated by θ and $\Pi \setminus \Pi_I$, $\overline{\Phi(V_I)} = (\cup_{b \in \mathbb{R}^+} S_b^\pm) \cup \Phi(\Pi \setminus \Pi_I)$. Therefore, we obtain that $\Phi(V_I) = \overline{\Phi(V_I)}$ if and only if $d = 1$. It remains to prove (1). Let B be the basis of $\overline{\Phi(V_I)}$ such that the set of positive roots of $\overline{\Phi(V_I)}$ with respect to B is $\overline{\Phi(V_I)} \cap \Phi^+$. Then it is clear that B must contain $\Pi \setminus \Pi_I$ and hence any minimal root in S_d . It follows that S_d has a minimum α and that $B = \{\alpha\} \cup \Pi \setminus \Pi_I$. \square

In the proof of the following proposition, we detail the explicit construction of the parabolic closures in the few cases in which $\Phi(V_I)$ is not parabolic.

Proposition 3.4. *All the root subsystems that occur in Table 1 are parabolic subsystems of Φ , except the second one of the B_n case, the first one of E_8 , and the one of F_4 . In these cases $\overline{\Phi(V_I)}$ is of type B_{n-1} , E_7 , and B_3 , respectively.*

Proof. Apart three exceptions, the diagrams in Table 1 are obtained by removing nodes of indexes i, j such that $\gcd(m_i, m_j) = 1$, thus correspond to parabolic subsystems, by Lemma 3.3. The three exceptions are obtained when removing nodes of indexes: $i = n-1, j = n$ in type B_n ; $i = 1, j = 3$ in type E_8 ; $i = 3, j = 4$ in type F_4 , in all cases being $\gcd(m_i, m_j) = 2$. By Lemma 3.3, in these cases a basis for $\overline{\Phi(V_I)}$ can be obtained from Π by removing α_i and α_j and adding the minimal root α such that $c_i(\alpha) = \frac{m_i}{2}$, $c_j(\alpha) = \frac{m_j}{2}$. Explicitly, $\alpha = \alpha_{n-1} + \alpha_n$

in case B_n , $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$ in case E_8 , and $\alpha = \alpha_2 + 2\alpha_3 + \alpha_4$ in case F_4 . This produces diagrams of type B_{n-1} , E_7 , and B_3 , respectively (the corresponding subsystems $\Phi(V_I)$ are of type D_{n-1} , A_7 , and A_3). \square

In Table 2, for each Φ we list the standard parabolic subsystems corresponding to the $(n-2)$ -faces and specify the orbit $[\omega_k^\perp]$ of the hyperplane they span. Thus, \mathcal{H}_Φ is the union of the $[\omega_k^\perp]$ that occur in the table.

The listed orbits are distinct, if not explicitly noticed.

In all cases except A_n and D_n , the Dynkin diagram of $\overline{\Phi(V_F)}$ determines uniquely the orbit of its hyperplane. The cases A_n and D_n require a (brief) direct check. For type A_n , it suffices to notice that there exist exactly two standard parabolic subsystems of rank $n-1$, which are of type A_{n-1} and can be transformed each into the other by W . In type D_n , the two standard parabolic faces obtained for $I = \{1, n\}$ and $I = \{1, n-1\}$ give two parabolic subsystems of type A_{n-1} : these can be transformed each into the other if and only if n is odd.

4. THE FACETS OF \mathcal{P}_Φ AND THE REGIONS OF \mathcal{H}_Φ

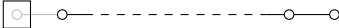






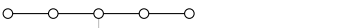
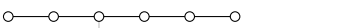
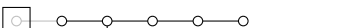
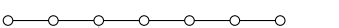

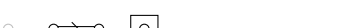

As already noticed in Proposition 3.1, for all irreducible root system Φ , the cones on the maximal faces of \mathcal{P}_Φ are unions of regions of the hyperplane arrangement \mathcal{H}_Φ . For types A and C , actually the cones on the maximal faces of \mathcal{P}_Φ are precisely the regions of the hyperplane arrangement \mathcal{H}_Φ . On the contrary, for types B and D , this is not the case and there are hyperplanes of \mathcal{H}_Φ intersecting the interior of some facets of \mathcal{P}_Φ . In this section, we show the different occurring phenomena.

First, we consider the case of type A . So let Φ be of type A_n and omit the subscript Φ anywhere. If $\alpha_1, \dots, \alpha_n$ are the simple roots, then the positive roots are all the sums $\sum_{k \in [i, j]} \alpha_k$, with $1 \leq i \leq j \leq n$. To simplify notation, we set $\alpha_{i, j} := \sum_{h=i}^j \alpha_h$, for all $1 \leq i \leq j \leq n$.

As noticed in Table 2, the arrangement \mathcal{H} is given by the orbit $[\check{\omega}_1^\perp]$ of the hyperplane orthogonal to the first fundamental coweight $\check{\omega}_1$ (which coincides with the orbit $[\check{\omega}_n^\perp]$ of the hyperplane orthogonal to the n -th fundamental coweight $\check{\omega}_n$).

Since the stabilizer of $\check{\omega}_1$ in the Weyl group W (contragredient representation) is the parabolic subgroup $W\langle\Pi \setminus \{\alpha_1\}\rangle$, the orbit of $\check{\omega}_1$ is obtained by acting with the set of the minimal coset representatives $W^1 := \{s_i \cdots s_1 \mid i = 0, \dots, n\}$ (for simplicity, we write s_k instead of s_{α_k} , for $k \in [n]$). For $i \in [1, n+1]$, let

TABLE 2. Parabolic subsystems corresponding to \mathcal{H}_Φ

A_n	 $[\omega_1^\perp] = [\omega_n^\perp]$	
B_n	 $[\omega_n^\perp]$	 $[\omega_1^\perp] \quad n \geq 4$
C_n	 $[\omega_1^\perp]$	
D_n	 $[\omega_{n-1}^\perp], [\omega_n^\perp]$	 $[\omega_1^\perp]$
	$[\omega_n^\perp] = [\omega_{n-1}^\perp]$ if and only if n is odd	
E_6	 $[\omega_1^\perp] = [\omega_6^\perp]$	 $[\omega_2^\perp]$
E_7	 $[\omega_2^\perp]$	 $[\omega_1^\perp]$
E_8	 $[\omega_2^\perp]$	 $[\omega_8^\perp]$
F_4	 $[\omega_4^\perp]$	
G_2	 $[\omega_1^\perp]$	

$\check{\omega}_{1,i} := s_{i-1} \cdots s_1(\check{\omega}_1)$. We have that

$$\check{\omega}_{1,i} = \begin{cases} \check{\omega}_1, & \text{if } i = 1, \\ \check{\omega}_i - \check{\omega}_{i-1}, & \text{if } i \notin \{1, n+1\}, \\ -\check{\omega}_n, & \text{if } i = n+1. \end{cases}$$

The hyperplanes of \mathcal{H} are exactly the hyperplanes generated by the sets of roots $(\Pi \cup \{-\theta\}) \setminus \{\alpha_i, \alpha_{i+1}\}$, for $i = 0, \dots, n$, where we intend $\alpha_0 = \alpha_{n+1} = -\theta$ (by Theorem 2.3, we already knew that \mathcal{H} should contain such hyperplanes, for $i = 1, \dots, n-1$).

By reasons of cardinality, the Weyl group W acts on \mathcal{H} as its group of permutations. The one-dimensional subspaces of $\mathcal{L}(\mathcal{H})$ are exactly the spaces $\mathbb{R}\alpha$, for all $\alpha \in \Phi$. In fact, let $\mathcal{I}_k = \cap_{i=1}^k \check{\omega}_{1,i}^\perp$ and notice that, for $k = 1, \dots, n$, \mathcal{I}_k contains the simple roots α_j for $j > k$ and does not contain the simple roots α_j for $j \leq k$; in particular $\mathcal{I}_1 \supsetneq \dots \supsetneq \mathcal{I}_n$. This implies that $\check{\omega}_{1,1}, \dots, \check{\omega}_{1,n}$ are linearly independent, and that $\mathcal{I}_k = \text{Span}(\alpha_j \mid j > k)$. In particular $\mathcal{I}_{n-1} = \mathbb{R}\alpha_n$ whence, since \mathcal{H} is W -stable, $\mathbb{R}\alpha$ belongs to $\mathcal{L}(\mathcal{H})$ for all $\alpha \in \Phi$ and, since W is $(n-1)$ -fold transitive on \mathcal{H} , these are exactly the one-dimensional subspaces of $\mathcal{L}(\mathcal{H})$.

Consider the two open halfspaces determined by $\check{\omega}_{1,i}^\perp$:

- $\check{\omega}_{1,i}^+ = \{x \in V \mid (x, \check{\omega}_{1,i}) > 0\}$,
- $\check{\omega}_{1,i}^- = \{x \in V \mid (x, \check{\omega}_{1,i}) < 0\}$.

and the regions of \mathcal{H}

$$\bigcap_{j=1}^{n+1} \check{\omega}_{1,j}^{\sigma_j},$$

where $\sigma_j \in \{+, -\}$. The regions are not empty, except exactly the two ones with the σ_j either all equal to $+$, or all equal to $-$. In fact, these two sets are clearly void. On the other hand, by Proposition 3.1, the number of regions of \mathcal{H} is greater than the number of facets of the root polytopes which, by Theorem 2.3, is equal to

$$\sum_{i=1}^n [W : W\langle \Pi \setminus \alpha_i \rangle] = \sum_{i=1}^n \frac{(n+1)!}{(i)!(n+1-i)!} = 2^{n+1} - 2.$$

Hence the number of regions of \mathcal{H} and the number of facets of \mathcal{P} are equal to $2^{n+1} - 2$. Thus we have proved the following theorem.

Theorem 4.1. *If the root system Φ is of type A , the closures of the regions of the hyperplane arrangement \mathcal{H}_Φ coincide with the cones on the facets of the root polytope \mathcal{P}_Φ .*

If we set, for all $i \in [n]$, $M_i = \{\alpha \in \Phi^+ \mid (\alpha, \check{\omega}_i) > 0\}$ (see Section 5 for the role of the sets M_i , $i \in [n]$, in the theory of abelian ideals) and

$$R_i = \bigcap_{j=1}^{n+1} \check{\omega}_{1,j}^{\sigma_j}$$

with $\sigma_j = +$ for $1 \leq j \leq i$ and $\sigma_j = -$ otherwise, then

$$M_i \subseteq \overline{R_i}$$

and the standard parabolic facet F_i , which is the convex hull of M_i , is equal to $\overline{R_i} \cap \{x \mid (x, \check{\omega}_i) = 1\}$. Since $\text{Span } M_i = \text{Span } \Phi$, this implies that $\text{Span } R_i = \text{Span } \Phi$. Now the set of regions R_i , for all $i \in [n]$, under the action of W covers all the regions $\bigcap_{j=1}^{n+1} \check{\omega}_{1,j}^{\sigma_j}$ except the two empty ones.

We now compute the characteristic polynomial and the face polynomial of \mathcal{H} .

Proposition 4.2. *Let Φ be of type A_n . Then*

(1) *the characteristic polynomial of \mathcal{H} is*

$$\chi_{\mathcal{H}}(t) = (-1)^n \sum_{i=1}^n (1-t)^i,$$

(2) *the face polynomial of \mathcal{H} is*

$$f_{\mathcal{H}}(x) = 1 + \sum_{i=1}^n \binom{n+1}{i+1} (2^{i+1} - 2)x^i.$$

Proof. To prove the first equality, we provide a direct computation of the Möbius function of $\mathcal{L}(\mathcal{H})$.

Since \mathcal{H} has $n+1$ hyperplanes and any such n hyperplanes are linearly independent, $\mathcal{L}(\mathcal{H})$ is obtained from the boolean algebra of rank $n+1$ by replacing all the elements of both rank n and $n+1$ with a single top element (of rank n), the null space $\underline{0}$, whose Möbius function is $\mu(\underline{0}) = (-1)^n n$. Hence, if $k \geq 1$, the number of $x \in \mathcal{L}(\mathcal{H})$ of dimension k is $\binom{n+1}{k+1}$ and, for all such x , $\mu(x) = (-1)^{\text{codim } x}$. Thus we have:

$$\chi_{\mathcal{H}}(t) = (-1)^n n + \sum_{k=1}^n \binom{n+1}{k+1} (-1)^{n-k} t^k$$

and we get the assertion since

$$(-1)^n \sum_{i=1}^n (1-t)^i = (-1)^n \sum_{i=1}^n \sum_{k=0}^i \binom{i}{k} (-t)^k = (-1)^n \sum_{k=0}^n \sum_{i=k}^n \binom{i}{k} (-t)^k$$

$$\text{and } \sum_{i=k}^n \binom{i}{k} = \begin{cases} \binom{n+1}{k+1}, & \text{if } k \neq 0, \\ n, & \text{if } k = 0. \end{cases}$$

To prove the second equality, consider the restricted hyperplane arrangement $(\mathcal{H} \setminus \{\check{\omega}_{1,n+1}^\perp\}) \cap \check{\omega}_{1,n+1}^\perp$: it is essentially the type A_{n-1} analogue of \mathcal{H} itself. In fact, this implies that the number of regions of this restricted arrangement is $2^n - 2$, and it is clear the same is true for all restricted arrangements $(\mathcal{H} \setminus \{\check{\omega}_{1,i}^\perp\}) \cap \check{\omega}_{1,i}^\perp$, $i \in [n+1]$. Hence we obtain that \mathcal{H} has $(n+1)(2^n - 2)$ faces of dimension $n-1$. Since \mathcal{H} has $n+1$ hyperplanes and any such n hyperplanes are linearly independent, by an analogous argument we obtain that \mathcal{H} has $\binom{n+1}{i} (2^{n+1-i} - 2)$

faces of dimension $n - i$, for $i = 0, \dots, n - 1$. Taking into account also the trivial face, we obtain the assertion. \square

By Theorem 2.5 and Proposition 4.2, (1), we re-obtain that $2^{n+1} - 2$ is the number of regions of \mathcal{H} .

We have the following.

Corollary 4.3. *The faces of \mathcal{P} are in natural bijection with the nontrivial faces of the hyperplane arrangement \mathcal{H} . Under this bijection, the k -dimensional faces of \mathcal{P} correspond to the $(k + 1)$ -dimensional faces of \mathcal{H} , for all $k = 0, \dots, n - 1$.*

Proof. The bijection between the facets of \mathcal{P} and the regions of \mathcal{H} (Theorem 4.1) is given by coning: for all $i \in [n]$, it maps F_i to R_i , which is the cone on F_i . This map induces the required bijection since, for all $K \subseteq [n]$, the intersection of the cones on F_k , for $k \in K$, is equal to the cone on the set $\cap_{k \in K} F_k$. In fact, this implies that the map sending $\cap_{k \in K} F_k$ to its cone is injective and surjective. \square

Corollary 4.4. *The face polynomial of \mathcal{P} is*

$$f_{\mathcal{P}}(x) = \sum_{i=0}^{n-1} \binom{n+1}{i+2} (2^{i+2} - 2) x^i.$$

Proof. By Corollary 4.3, the face polynomials of $f_{\mathcal{H}}(x)$ and $f_{\mathcal{P}}(x)$ satisfy the relation $f_{\mathcal{H}}(x) = 1 + x f_{\mathcal{P}}(x)$. Hence the assertion follows by Proposition 4.2, (2). \square

Each vertex of \mathcal{P} belongs to exactly 2^{n-1} facets of \mathcal{P} . This is clear if we observe that each vertex v belongs to exactly $n - 1$ hyperplanes of \mathcal{H} and, if i and i' are the two indices such that $(v, \check{\omega}_{1,i})$ and $(v, \check{\omega}_{1,i'})$ are nonzero, then these two values have different signs. It follows that the 2^{n-1} sign choices for $\{\check{\omega}_{1,j}^{\pm} \mid j \neq i, i'\}$ correspond exactly to the regions of \mathcal{H} whose closures contain v .

The realization of the standard parabolic facet F_i , $i \in [n]$, as the intersection of a closed region of \mathcal{H} with the affine hyperplane $(x, \check{\omega}_i) = 1$ shows that F_i has n facets for $i = 1$ and $i = n$ and at most $n + 1$ facets for $1 < i < n$, since $\check{\omega}_i^{\perp} \in \mathcal{H}$ if and only if $i = 1$ or $i = n$. Thus F_1 and F_n are $(n - 1)$ -simplices.

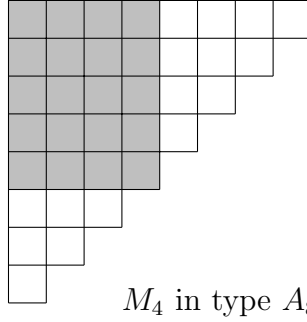
We recall the Young diagrams combinatorics of the positive roots of type A_n , which is useful for representing the abelian ideals and hence the faces. The A_n root diagram is a partial $n \times n$ matrix filled with the positive roots: the i -th row consists of the $n - i$ positions from $(i, 1)$ to $(i, n - i)$ (staircase tableau) and the position (i, j) is filled with the root $\alpha_{j, n-i}$.

As an example, the following is the A_4 root diagram:

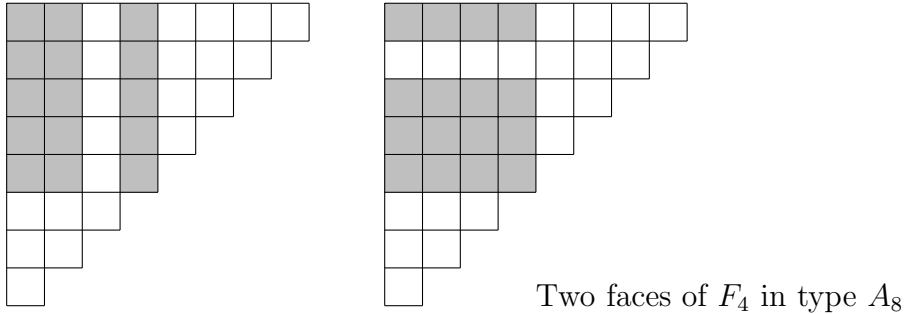
$\alpha_{1,4}$	$\alpha_{1,3}$	$\alpha_{1,2}$	α_1
$\alpha_{2,4}$	$\alpha_{2,3}$	α_2	
$\alpha_{3,4}$	α_3		
α_4			

The root order corresponds to the reverse order of the matrix positions, i.e, if β fills the position (i, j) and γ fills the position (h, k) , then $\beta \geq \gamma$ if and only if $i \leq h$ and $j \leq k$ (i.e., if and only if β is at the north-west of γ). We shall identify the A_n diagrams and subdiagrams with the corresponding sets of roots.

For $i = 1, \dots, n$, the standard parabolic facet F_i is the convex hull of M_i , the dual order ideal generated by the root α_i in the root poset of Φ , and M_i is exactly the set of vertices of F_i and form a maximal rectangle in the tableau.



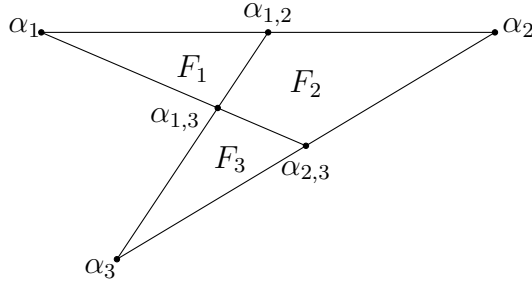
For $1 < i < n$, the facets of F_i are actually $n+1$ and are obtained intersecting with one of the hyperplanes $\check{\omega}_{1,j}^\perp$, $j \in [n+1]$. Hence they are of two kinds, according to whether $j \leq i$ or $j > i$: the first are obtained removing the corresponding column in the rectangle M_i , the second removing the corresponding row. It is immediate to see that these are isometric, respectively, to the facets of indices i and $i-1$ of the type A_{n-1} analogue of \mathcal{P} . Thus \mathcal{P} has a recursive structure, in the sense that its faces of dimension less than $n-1$ are the faces of the A_k analogues of \mathcal{P} , for $k < n$.



Moreover, this description of the facets of F_i implies that F_i is congruent to the product of two simplices, as already pointed out in [1] for the root polytope obtained in the usual coordinate description of Φ (settled, for example, in [4]). In particular, F_i is congruent to the product of a $(i-1)$ -simplex and a $(n-i)$ -simplex.

Recall from Theorem 2.3 that the facets of \mathcal{P} are obtained from F_1, \dots, F_n through the action of W , that these orbits are disjoint, and that the number of facets in the orbit of F_i is $[W : \text{Stab}_W(\check{\omega}_i)] = \binom{n+1}{i}$. On the other hand, the whole automorphism group of Φ joins the orbits WF_i and WF_{n+1-i} , for $i \in [n]$, so that, under this group, \mathcal{P} has exactly $\lceil \frac{n+1}{2} \rceil$ orbits of facets, and the i -th orbit has cardinality $2\binom{n+1}{i}$, for $i = 1, \dots, \lceil \frac{n+1}{2} \rceil$.

We briefly illustrate our results in type A_3 , when the root polytope is the well known “cuboctahedron”, that is the intersection of a cube with its dual octahedron (see [10]). This has fourteen facets, six of which are squares, corresponding to the standard parabolic facet F_2 , and eight regular triangles, corresponding to either the facet F_1 or the facet F_3 . In the next figure, we see a section of the cuboctahedron and how the four hyperplanes of \mathcal{H} cut it.



We now treat the case of the root systems of type C . The description of \mathcal{P}_Φ is much simpler and most of the combinatorics developed for type A has its analogue. Henceforward, we assume that Φ is a root system of type C_n and omit the subscript Φ anywhere. As noticed in [5], the root polytope of every root system is the convex hull of its long roots. In our case, the long positive roots are $\lambda_i := 2(\sum_{k=i}^{n-1} \alpha_k) + \alpha_n$, for $i \in [n]$, and form a (orthogonal) basis of $\text{Span } \Phi$. Then the root polytope \mathcal{P} is a hyperoctahedron (or cross-polytope). Its facets correspond one to one to the octants of the cartesian system given by the basis of the long roots.

As noticed in Table 2, the arrangement \mathcal{H} is given by the orbit $[\check{\omega}_1^\perp]$ of the hyperplane orthogonal to the first fundamental coweight $\check{\omega}_1$. Since the stabilizer of $\check{\omega}_1$ in the Weyl group W (contragredient representation) is the parabolic subgroup $W\langle \Pi \setminus \{\alpha_1\} \rangle$, the orbit of $\check{\omega}_1$ is obtained by acting with the set of the minimal coset representatives $W^1 := \{s_i \cdots s_1 \mid i = 0, \dots, n-1\} \cup \{s_i \cdots s_{n-1} s_n \cdots s_1 \mid i = 1, \dots, n\}$ (for simplicity, we write s_k instead of s_{α_k} , for $k \in [n]$). For $i \in [1, n]$, let

$\check{\omega}_{1,i} := s_{i-1} \cdots s_1(\check{\omega}_1)$. We have

$$\check{\omega}_{1,i} = \begin{cases} \check{\omega}_1, & \text{if } i = 1, \\ \check{\omega}_i - \check{\omega}_{i-1}, & \text{if } i \notin \{1, n\}, \\ 2\check{\omega}_n - \check{\omega}_{n-1}, & \text{if } i = n. \end{cases}$$

Moreover, $s_n(2\check{\omega}_n - \check{\omega}_{n-1}) = -(2\check{\omega}_n - \check{\omega}_{n-1})$ and hence the second n functional we obtain are the opposites of the first n . Thus the hyperplane of \mathcal{H} are the hyperplane orthogonal to $\check{\omega}_{1,i}$, for $i \in [n]$. Being the n hyperplanes linearly independent, the number of regions of \mathcal{H} is 2^n .

Hence, Proposition 3.1 implies the following theorem.

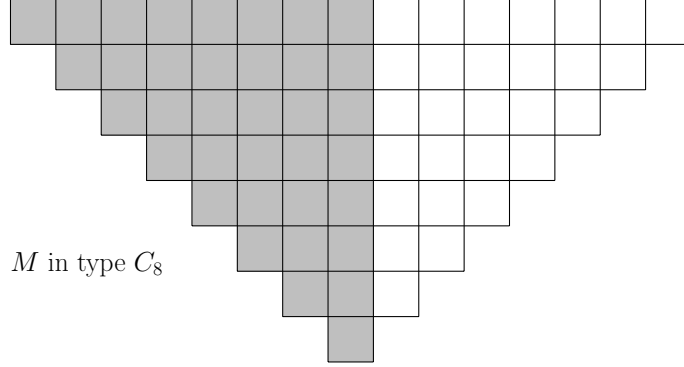
Theorem 4.5. *If the root system Φ is of type C , the closures of the regions of the hyperplane arrangement \mathcal{H}_Φ coincide with the cones on the facets of the root polytope \mathcal{P}_Φ .*

We recall the Young diagrams combinatorics for the root system of type C_n . To simplify notation, we set $\alpha_{i,j} := \sum_{h=i}^j \alpha_h$, for all $1 \leq i \leq j \leq n$. The C_n root diagram is a partial $n \times 2n - 1$ matrix filled with the positive roots: the i -th row consists of the $2(n - i) + 1$ positions from (i, i) to $(i, 2n - i)$; the positions (i, j) , with $i \leq j < n$, are filled with the roots $\alpha_{j,n-1} + \alpha_{i,n}$, the positions (i, j) , with $i \leq j = n$, are filled with the roots $\alpha_{i,n}$, and the positions (i, j) , with $n + 1 \leq j \leq 2n - i$, are filled with the roots $\alpha_{i,2n-j}$. Note that the long positive roots are in positions (i, i) , $i \in [n]$. As an example, the following is the C_4 root diagram:

$\alpha_{1,3} + \alpha_{1,4}$	$\alpha_{2,3} + \alpha_{1,4}$	$\alpha_3 + \alpha_{1,4}$	$\alpha_{1,4}$	$\alpha_{1,3}$	$\alpha_{1,2}$	α_1
	$\alpha_{2,3} + \alpha_{2,4}$	$\alpha_3 + \alpha_{2,4}$	$\alpha_{2,4}$	$\alpha_{2,3}$	α_2	
		$\alpha_3 + \alpha_{3,4}$	$\alpha_{3,4}$	α_3		
			α_4			

As for type A_n , the root order corresponds to the reverse order of the matrix positions, i.e, if β fills the position (i, j) and γ fills the position (h, k) , then $\beta \geq \gamma$ if and only if $i \leq h$ and $j \leq k$. We shall identify the C_n diagrams and subdiagrams with the corresponding sets of roots.

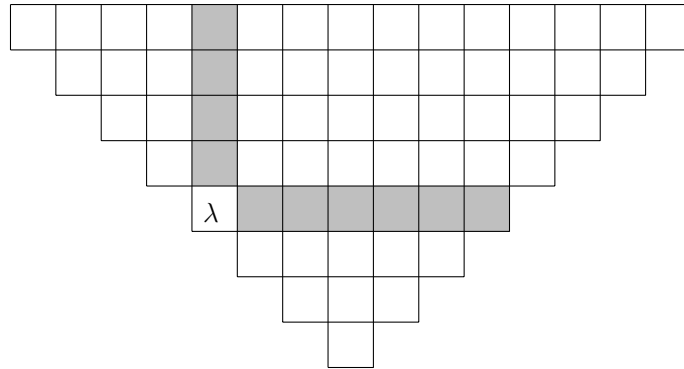
In type C_n , there is a unique standard parabolic facet F_n and this is the convex hull of the unique maximal abelian ideal $M := M_n = \{\alpha \in \Phi^+ \mid (\alpha, \check{\omega}_n) > 0\}$ (see Section 5), which is the dual order ideal generated by the unique long simple root α_n . This is the convex polytope having exactly the positive long roots as its set of vertices.



In the diagram representation of Φ^+ and M , the facets of F are obtained intersecting with one of the hyperplanes λ_k^\perp orthogonal to the long root λ_k , $k \in [n]$, and hence are the convex hulls of the sets obtained by removing from M the roots that are also in $U_{\lambda_k} \cup R_{\lambda_k} \cup \{\lambda_k\}$, where, for any long positive root λ , we have set

$$(4.1) \quad \begin{aligned} U_\lambda &:= \{\beta \in \Phi^+ \mid \beta - \lambda \in \Phi^+\} \\ R_\lambda &:= \{\beta \in \Phi^+ \mid \lambda - \beta \in \Phi^+\} \end{aligned}$$

In the diagram representation of the root poset, the roots in U_λ and R_λ are obtained from λ , respectively, going up and to the right.



For Φ of type B_n or D_n , $n \geq 4$, the lattice of regions of \mathcal{H} is strictly finer than the fan associated to \mathcal{P} . In fact, some hyperplanes in \mathcal{H} cut some facets of \mathcal{P} into two nontrivial parts. We show this for \mathcal{P} of type B_n ; since the embedded D_n made of the long roots of B_n has the same polytope, we obtain the analogous result also for D_n . So let Φ be of type B_n , $n \geq 4$. From Table 2, we see that $\check{\omega}_1^\perp$ belongs to \mathcal{H} . Recalling the extended Dynkin diagram of C_n (the dual root

system), we see that $\check{\omega}_1$ is parallel to the highest short root θ_s of Φ . Since the short positive roots in Φ form an orthogonal basis of $\text{Span } \Phi$, the orbit of $\check{\omega}_1$ consists of the vectors of an orthogonal basis of $\text{Span } \Phi$ together with their opposites. This implies, in particular, that $\mathbb{R}\check{\omega}_1$ belongs to the intersection lattice of \mathcal{H} , being the intersection of the $n - 1$ hyperplanes other than $\check{\omega}_1^\perp$ in the orbit $[\check{\omega}_1^\perp]$. But from [5, Proposition 3.3], we now that $\mathbb{R}\check{\omega}_1$ contains the barycenter of the standard parabolic facet F_1 of \mathcal{P} , therefore each of the hyperplanes in $[\check{\omega}_1^\perp] \setminus \check{\omega}_1^\perp$ cuts F_1 in two nontrivial parts.

5. PRINCIPAL MAXIMAL ABELIAN IDEALS OF THE BOREL SUBALGEBRA

In this section, we first provide some general results on principal abelian ideals that hold for all irreducible root systems. Then we shall restrict our attention to the types A_n and C_n . In these cases, the principal abelian ideals corresponding to the standard parabolic facets are exactly the maximal abelian ideals. In this cases, we will construct a triangulation of the standard parabolic facets strictly related to the poset of the abelian ideals. In this section, we state the results, which are formally the same for both types. The proofs will be given, separately for the two types, in Sections 6 and 8.

For any positive root α , we denote by M_α the dual order ideal of the root poset generated by α :

$$M_\alpha = \{\beta \in \Phi^+ \mid \beta \geq \alpha\};$$

for $i \in [n]$, we set $M_i = M_{\alpha_i}$, so that

$$M_i = \{\beta \in \Phi^+ \mid (\beta, \check{\omega}_i) > 0\}.$$

Recall that we denote by m_α , $\alpha \in \Pi$, the coordinates of θ and we set $m_i = m_{\alpha_i}$, i.e. $m_i = (\theta, \check{\omega}_i)$.

Lemma 5.1. *The principal ad-nilpotent ideal M_i is abelian if and only if $m_i = 1$.*

Proof. The statement of the lemma is equivalent to the assertion that M_i is abelian if and only if every positive root has 0 or 1 as i -th coordinate with respect to the basis given by the simple roots. Clearly, if every positive root has i -th coordinate equal to 0 or 1, then the sum of two roots in M_i cannot be a root, and hence M_i is abelian. Conversely, by contradiction, let β be a minimal root in M_i with i -th coordinate ≥ 2 : take a simple root α such that $\beta - \alpha \in \Phi^+$. By minimality, $\alpha = \alpha_i$ and both α_i , $\beta - \alpha_i$ are in M_i . This is a contradiction since M_i is abelian. \square

For instance, in type A_n , all ideals M_i are abelian while, in type C_n , only i_{M_n} is.

Lemma 5.2. *If M_i is abelian, then it is a maximal abelian ideal.*

Proof. Let M_i be abelian and let γ be a maximal positive root not in M_i . Let $\beta \triangleright \gamma$. Hence $\beta \in M_i$ by maximality and, since the root poset is ranked by the height, $\beta = \gamma + \alpha$ for a certain simple root α . Since $\gamma \notin M_i$ and $\beta \in M_i$, we have that $\alpha = \alpha_i$. Thus we get the assertion since there is no abelian ideal containing both γ and α_i since their sum is a root. \square

We set

$$W^i = \{w \in W \mid D_r(w) \subseteq \{\alpha_i\}\}.$$

It is well known that W^i is the set of minimal length representatives of the left cosets $W/W\langle \Pi \setminus \{\alpha_i\} \rangle$.

Lemma 5.3. *Let $w \in W$. Then $w \in W^i$ if and only if $\overline{N}(w) \subseteq M_i$.*

Proof. The only simple root in M_i is α_i , hence the claim follows directly from property (2.1) and equation (2.3). \square

Proposition 5.4. *Let $i \in [n]$ be such that $m_i = 1$, and $N \subseteq M_i$. Then $M_i \setminus N$ is an abelian ideal if and only if there exists $w \in W^i$ such that $N = \overline{N}(w)$.*

Proof. By Lemma 5.1, M_i is an abelian ideal: in particular the sum of any two roots in M_i is not a root.

Assume that $M_i \setminus N$ is an abelian ideal. By Lemma 5.3, it suffices to prove that $N = \overline{N}(w)$ for some $w \in W$ or, equivalently, that N and $\Phi^+ \setminus N$ are both closed. Clearly, N is closed because it is contained in M_i . Suppose $\beta, \beta' \in \Phi^+ \setminus N$. If both are in $M_i \setminus N$, by the same argument as before $\beta + \beta'$ is not a root. If both do not belong to M_i , then also $\beta + \beta'$ does not belong to M_i since its i -th coordinate is 0, and hence it belongs to $\Phi^+ \setminus N$. If one of the root, say β , is in $M_i \setminus N$, the other is not in M_i and $\beta + \beta'$ is a root, then $\beta + \beta' \in M_i$ since its i -th coordinate is 1, and hence $\beta + \beta'$ belongs to $M_i \setminus N$ since this set is a dual order ideal.

Conversely, assume that $N = \overline{N}(w)$ for some $w \in W^i$, and let $\beta \in M_i \setminus N$, $\beta' > \beta$. Then $\beta' - \beta$ is sum of positive roots not in M_i , since $(\beta, \check{\omega}_i) = (\beta', \check{\omega}_i) = 1$. Hence β' is a positive linear combination of roots in $\Phi^+ \setminus N$ and thus belongs to $\Phi^+ \setminus N$ since this is a convex set. \square

Proposition 5.5. *Fix $i \in [n]$ such that $m_i = 1$. Let $w \in W^i$, $\gamma, \gamma' \in M_i$, and $\gamma' \leq \gamma$. Then $w(\gamma') \leq w(\gamma)$.*

Proof. If $\gamma, \gamma' \in M_i$ and $\gamma' \leq \gamma$, then $\gamma - \gamma'$ is a sum of simple roots different from α_i . If $w \in W^i$, w maps all simple roots other than α_i to positive roots, hence $w(\gamma - \gamma') = w(\gamma) - w(\gamma')$ is a sum of positive roots, i.e. $w(\gamma') \leq w(\gamma)$. \square

Remark 5.1. By [6], Remark 7.3, if $m_\alpha = 1$, then $\max \mathcal{I}_{ab}(\alpha) = M_\alpha$. In particular, for all $I \in \mathcal{I}_{ab}(\alpha)$, $I \subseteq M_\alpha$. Then, by Theorem 2.1, in types A and C , the abelian ideals of the form M_α are the unique maximal abelian ideals.

The above facts together with Theorem 2.3 imply directly the following result.

Theorem 5.6. *If Φ is of type A_n or C_n , the set of the standard parabolic facets of \mathcal{P}_Φ is the set of convex hulls of the maximal abelian ideals of Φ^+ .*

In Sections 6 and 8, we construct a triangulation \mathcal{T} of the polytope \mathcal{P}_Φ for Φ of type A or C , which is strictly related to the poset of abelian ideals.

In order to obtain a triangulation of \mathcal{P}_Φ , we construct a triangulation of the standard parabolic facets and transport it to the whole polytope by the action of W . For the types A_n or C_n , we should provide a triangulation of the facets F_α , for all long simple α .

For each abelian ideal I in Φ^+ , we define the *border* $B(I)$ of I as follows:

$$(5.1) \quad B(I) = \{\beta \in I \mid \beta - \sum_{\substack{\alpha \in \Phi^+ \cup \{0\} \\ \beta - \alpha \in \Pi}} (\beta - \alpha) \notin I\}.$$

Theorem 5.7. *Let Φ be of type A_n or C_n and let α be a long simple root in Π . Then:*

- (1) *for all ideals $I \subseteq M_\alpha$, $\dim(\text{Span } I) = n$ if and only if $I \in \mathcal{I}_{ab}(\alpha)$;*
- (2) *for all $I \in \mathcal{I}_{ab}(\alpha)$, $B(I)$ is a basis of the root lattice;*
- (3) *for all $I \in \mathcal{I}_{ab}(\alpha)$, $\{\text{Conv}(B(J)) \mid J \in \mathcal{I}_{ab}(\alpha) \text{ and } J \subseteq I\}$ is a triangulation of $\text{Conv}(I)$.*

In particular, $\{\text{Conv}(B(I)) \mid I \in \mathcal{I}_{ab}(\alpha)\}$ is a triangulation of F_α .

Theorem 5.7 will be proved in Section 6 for type A and in Section 8 for type C .

Remark 5.2. If we view \widehat{W} as a group of affine transformation of the Euclidean space in the usual way [4], then \widehat{W}_{ab} is the set of all elements that map the fundamental alcove \mathcal{A} into $2\mathcal{A}$ [8]. Moreover, if I is an abelian ideal, then there exists $\alpha \in \Pi$ such that $I \in \mathcal{I}_{ab}(\alpha)$ if and only if $w_I(\mathcal{A})$ has a facet on the affine hyperplane $H_{\theta,2} := \{x \mid (x, \theta) = 2\}$. Indeed, the facet of \mathcal{A} orthogonal to α must be mapped by w into $H_{\theta,2}$. If $m_\alpha = 1$, then the facet of \mathcal{A} orthogonal to α has the same measure of the facet of \mathcal{A} orthogonal to θ , since there is an element of the extended affine Weyl group that maps one facet into the other ([19], see also [7]). Therefore, since in the A_n and C_n cases $m_\alpha = 1$ for all the long simple roots α , we obtain that $|\cup_{\alpha \in \Pi_\ell} \mathcal{I}_{ab}(\alpha)| = 2^{n-1}$. It follows that the total number of simplices that occur in the triangulations of the fundamental facets of \mathcal{P} is 2^{n-1} , in both types.

For all long simple roots α_i , we set

$$\mathcal{T}_i = \{ \text{Conv}_0(B(I)) \mid I \in \mathcal{I}_{ab}(\alpha_i) \}.$$

Since the stabilizer of the facet F_i is the standard parabolic subgroup $W\langle \Pi \setminus \{\alpha_i\} \rangle$, if for all $\alpha_i \in \Pi_\ell$ we choose any set \mathcal{R}_i of representatives of the left cosets of $W\langle \Pi \setminus \{\alpha_i\} \rangle$ in W and set

$$\mathcal{R}_i \mathcal{T}_i = \{ wT \mid w \in \mathcal{R}_i, T \in \mathcal{T}_i \},$$

then

$$\bigcup_{\alpha_i \in \Pi_\ell} \mathcal{R}_i \mathcal{T}_i$$

is a triangulation of \mathcal{P}_Φ .

Theorem 5.8. *For all $\alpha_i \in \Pi_\ell$, let W^i be the set of minimal length representatives of the left cosets of $W/W\langle \Pi \setminus \{\alpha_i\} \rangle$ and set*

$$\mathcal{T} = \bigcup_{\alpha_i \in \Pi_\ell} W^i \mathcal{T}_i, \quad \mathcal{T}^+ = \{ T \in \mathcal{T} \mid T \subset \mathcal{P}^+ \}$$

Then \mathcal{T} is a triangulation of \mathcal{P} and \mathcal{T}^+ is a triangulation of \mathcal{P}^+ . In particular, $\mathcal{P}^+ = \mathcal{P} \cap \mathcal{C}^+$, where \mathcal{C}^+ is the positive cone generated by Φ^+ .

It is clear that, in general, the action of the stabilizer of a facet F does not preserve a fixed triangulation of F . The choice of the minimal length representatives is essential in Theorem 5.8.

6. TRIANGULATION OF \mathcal{P} : TYPE A

Along this section, Φ is a root system of type A_n . Recall that the positive roots are the roots $\alpha_{i,j} := \sum_{h=i}^j \alpha_h$, for all $1 \leq i \leq j \leq n$.

We give here an elementary proof that the M_i are all the maximal abelian ideals of Φ . It is clear that an abelian ideal cannot contain any pair of roots $\sum_{k \in [i,j]} \alpha_k$ and $\sum_{k \in [i',j']} \alpha_k$ with disjoint $[i, j]$ and $[i', j']$. Hence the result follows by the following easy Erdős-Ko-Rado type result.

Proposition 6.1. *Let $S = \{[i, j] \mid 1 \leq i \leq j \leq n\}$ be the set of all intervals in $[n]$ and, for $i \in [n]$, let S_i be the subset of all intervals containing i . Let $U \subseteq S$ be maximal among the subsets containing pairwise intersecting intervals. Then there exists $i \in [n]$ such that $U = S_i$.*

Proof. By contradiction, assume $U \not\subseteq \{S_i \mid i \in [n]\}$. By maximality, if $[i, j] \in U$, then U contains every interval containing $[i, j]$. Since $U \neq S_1$, there exists $k \in [n]$, $k \neq 1$, such that $[k, n] \in U$: let $\bar{k} = \max\{k \mid [k, n] \in U\}$. Then all subintervals of $[\bar{k}, n]$ in U must contain \bar{k} , and since $U \neq S_{\bar{k}}$, we obtain that $[1, \bar{k} - 1] \in U$. Thus we get a contradiction since $[1, \bar{k} - 1] \cap [\bar{k}, n] = \emptyset$. \square

In the following proposition, we specialize the result of Proposition 2.2 and determine the set $\mathcal{I}_{ab}(\alpha_i)$, for all $i \in [n]$.

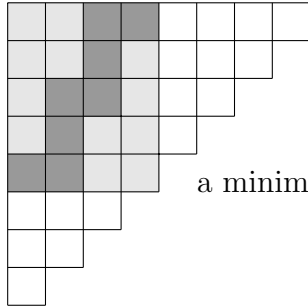
Proposition 6.2. *Let I be an abelian ideal and $i \in [n]$. Then $I \in \mathcal{I}_{ab}(\alpha_i)$ if and only if $I \subseteq M_i$ and $\{\alpha_{1,i}, \alpha_{i,n}\} \subseteq I$.*

Proof. Let I be an abelian ideal in $\mathcal{I}_{ab}(\alpha_i)$. As we have already noticed, $I \subseteq M_i$, since M_i is the maximum of $\mathcal{I}_{ab}(\alpha_i)$. On the other hand, $I \not\subseteq M_j$, for $i \neq j$, since the elements in $\mathcal{I}_{ab}(\alpha_i)$ and those in $\mathcal{I}_{ab}(\alpha_j)$ are pairwise incomparable for $i \neq j$.

It remains to prove that, for all abelian ideals I contained in M_i , $w_I^{-1}(-\theta + 2\delta) = \alpha_i$ if and only if $\{\alpha_{1,i}, \alpha_{i,n}\} \subseteq I$.

For all $\beta, \gamma \in \Phi^+$, $\beta + \gamma = \theta$ if and only if there exists $j \in [n-1]$ such that $\beta = \alpha_{1,j}$ and $\gamma = \alpha_{j+1,n}$, or vice versa. Since $\alpha_{1,j} \geq \alpha_{1,i}$ for $i \leq j$ and $\alpha_{j+1,n} \geq \alpha_{i,n}$ if $i > j$, it follows that if $\{\alpha_{1,i}, \alpha_{i,n}\} \subseteq I$, then for all $\beta, \gamma \in \Phi^+$ such that $\beta + \gamma = \theta$, exactly one of β, γ belongs to I . Conversely, if $\{\alpha_{1,i}, \alpha_{i,n}\} \not\subseteq I$, there exists at least one decomposition of θ as a sum of positive roots β and γ with $\beta \notin I$ and $\gamma \notin I$. Hence the claim follows from Proposition 2.2. \square

For all $\alpha, \beta \in \Phi^+$, by a *path from α to β* we mean a sequence $(\alpha = \beta_1, \beta_2, \dots, \beta_k = \beta)$ such that, for $i = 1, \dots, k-1$, β_i covers, or is covered by, β_{i+1} in the root poset, i.e. $\beta_i - \beta_{i+1} \in \pm\Pi$. For convenience sake, we will sometimes identify a path with its underlying set. A *minimal path from α to β* is a path of minimal length among all paths from α to β .



a minimal path from $\alpha_{1,4}$ to $\alpha_{4,8}$ in type A_8

The minimal paths from $\alpha_{1,i}$ to $\alpha_{i,n}$ are in natural bijection with the abelian ideals that contain both $\alpha_{1,i}$ and $\alpha_{i,n}$, i.e. with the elements in $\mathcal{I}_{ab}(\alpha_i)$. The bijection associates to each minimal path B the dual order ideal $I(B)$ that it generates in Φ^+ , i.e.

$$I(B) := \{\alpha \in \Phi^+ \mid \alpha \geq \beta \text{ for some } \beta \in B\}$$

and, conversely, to each abelian ideal I containing $\alpha_{1,i}$ and $\alpha_{i,n}$ its *border*

$$B(I) = \{\alpha_{s,t} \in I \mid \alpha_{s+1,t-1} \notin I\}.$$

This is the specialization to A_n of the definition given in (5.1).

It is clear that, for all $i \in [n]$, any minimal path B from $\alpha_{1,i}$ to $\alpha_{i,n}$ contains n roots. Moreover, the roots in B are linearly independent, since the set of differences between two adjacent roots is $\Pi \setminus \{\alpha_i\}$, and any root in B has coefficient 1 in α_i . Thus, the roots in B are a basis of $\text{Span } \Phi$. Since Φ is of type A_n , this implies that they are \mathbb{Z} -basis of the root lattice. This is a well known fact; we prove it here for completeness.

Proposition 6.3. *Every vector basis of $\text{Span } \Phi$ contained in Φ is a basis of the root lattice.*

Proof. We proceed by induction on n , the case $n = 1$ being trivial. Let $n > 1$ and β_1, \dots, β_n be linearly independent vectors in Φ : we may clearly assume that they are in Φ^+ . Suppose that there is a unique $i \in [n]$ such that $\beta_i > \alpha_n$ (i.e. $c_n(\beta_i) \neq 0$): then, by induction hypothesis, $\{\beta_1, \dots, \beta_n\} \setminus \{\beta_i\}$ gives a basis of the lattice generated by $\Pi \setminus \{\alpha_n\}$. Since $c_n(\beta_i) = 1$, we get the assertion.

Suppose now that the vectors of the basis which are greater than α_n are $\beta_{i_1}, \dots, \beta_{i_s}$, with $s > 1$ and $\beta_{i_1} > \beta_{i_2} > \dots > \beta_{i_s}$. Then $\{\beta_t \mid t \in [n], t \neq i_1, \dots, i_{s-1}\} \cup \{\beta_t - \beta_{i_s} \mid t \in \{i_1, \dots, i_{s-1}\}\} \subseteq \Phi^+$ is a vector basis generating the same lattice but with a unique element greater than α_n . Hence we may conclude applying the previous argument. \square

The set $\mathcal{I}_{ab}(\alpha_i)$ parametrizes a triangulation of F_i .

Proposition 6.4. *Let $i \in [n]$. The set*

$$\mathcal{T}'_i = \{\text{Conv}(B(I)) \mid I \in \mathcal{I}_{ab}(\alpha_i)\}$$

is a triangulation F_i . Two simplices are adjacent in \mathcal{T}'_i if and only if the corresponding abelian ideals differ for only one element.

In particular, the number of simplices of every root-based triangulation of the standard parabolic facet F_i equals the cardinality $\mathcal{I}_{ab}(\alpha_i)$.

Proof. By Proposition 6.3, all root-based triangulations have the same number of simplices. Moreover, as already noticed, every abelian ideal in $\mathcal{I}_{ab}(\alpha_i)$ is uniquely determined by its border. Hence the first statement implies the second one.

To prove the first statement, recall that the standard parabolic facet F_i is congruent to the product of two simplices and that the borders of the ideals we are considering coincides with the minimal paths from $\alpha_{1,i}$ to $\alpha_{i,n}$ in the rectangle M_i corresponding to F_i . The triangulation \mathcal{T}_i is the so called “staircase triangulation” of the product of two simplices (see [12], §6.2.3) and satisfies the required property. \square

The stabilizer in W of the face F_i of \mathcal{P} is the parabolic subgroup $W\langle\Phi \setminus \alpha_i\rangle$. Let W^i be the set of the minimal length representatives of its left cosets, which

corresponds to the orbit of F_i under W . Through the action of the elements in W^i , for all $i \in [n]$, we can induce, from the triangulation of the standard parabolic facets F_i , a triangulation of the whole \mathcal{P} . Hence we obtain the following result.

Theorem 6.5. *For all $i \in [n]$, let*

$$\mathcal{T}_i = \{ \text{Conv}_0(B(I)) \mid I \in \mathcal{I}_{ab}(\alpha_i) \} \quad \text{and} \quad \mathcal{T} = \bigcup_{i \in [n]} W^i \mathcal{T}_i.$$

Then \mathcal{T} is a triangulation of \mathcal{P} .

The triangulation \mathcal{T} of \mathcal{P} is parametrized by the set

$$\{(i, w, T) \mid i \in [n], w \in W^i, T \in \mathcal{T}_i\}.$$

In particular, as already noticed in [1], \mathcal{T} has $\sum_{i=1}^n |W^i| |\mathcal{T}_i| = \sum_{i=1}^n \binom{n+1}{i} \binom{n-1}{n-i} = (n+1)C_n$ simplices, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number

Actually, the triangulation \mathcal{T} induces a triangulation of the positive root polytope \mathcal{P}^+ .

Theorem 6.6. *Let \mathcal{C}^+ denote the positive cone generated by Π . Then $\mathcal{P}^+ = \mathcal{C}^+ \cap \mathcal{P}$ and the triangulation \mathcal{T} of \mathcal{P} restricts to a triangulation of \mathcal{P}^+ .*

Proof. We shall prove the following two statements, which give the result:

- (1) the triangulation \mathcal{T} restricts to a triangulation of $\mathcal{C}^+ \cap \mathcal{P}$,
- (2) $\mathcal{P}^+ = \mathcal{C}^+ \cap \mathcal{P}$.

The first assertion is equivalent to requiring that every simplex of the triangulation \mathcal{T} is either contained in \mathcal{C}^+ , or intersects it in a null set. Hence, we must show that, for any $i \in [n]$, $w \in W^i$, and $T \in \mathcal{T}_i$, either $w(T) \subseteq \mathcal{C}^+$, or $w(T) \cap \mathcal{C}^+$ has volume equal to 0.

Let $i \in [n]$, $w \in W^i$, $T \in \mathcal{T}_i$, $T = \text{Conv}_0(B(I))$ with I abelian ideal in M_i , such that $w(T) \not\subseteq \mathcal{C}^+$. Then, $w(B(I)) \not\subseteq \Phi^+$, hence there exists $\gamma \in B(I)$ and $t \in [n]$ such that $(w(\gamma), \check{\omega}_t) < 0$. We need to prove that, for all $\gamma' \in B(I)$, we have that $(w(\gamma'), \check{\omega}_t) \leq 0$ (so that the hyperplane $\check{\omega}_t^\perp$ separates $w(T)$ and \mathcal{C}^+). If $\gamma' \leq \gamma$, this follows from Proposition 5.5. So we assume $\gamma' \not\leq \gamma$. Then there exists $\gamma'' \in M_i$ such that $\gamma'' - \gamma \in \Phi^+$ and $\gamma'' - \gamma' \in \Phi^+ \cup \{0\}$. By Proposition 5.5, it suffices to prove that $(w(\gamma''), \check{\omega}_t) \leq 0$. Since both γ and γ'' are in M_i , $\gamma'' - \gamma \notin M_i$ and hence $w(\gamma'' - \gamma) \in \Phi^+$. It follows that $(w(\gamma''), \check{\omega}_t) = (w(\gamma'' - \gamma), \check{\omega}_t) + (w(\gamma), \check{\omega}_t) = (w(\gamma'' - \gamma), \check{\omega}_t) - 1 \leq 0$, since $m_t = 1$ for all $t \in [n]$ and thus $|\langle \alpha, \check{\omega}_t \rangle| \leq 1$ for all $\alpha \in \Phi$.

It remains to prove the second assertion. The inclusion $\mathcal{P}^+ \subseteq \mathcal{C}^+ \cap \mathcal{P}$ is obvious, since the origin and positive roots are contained in both the convex sets \mathcal{C}^+ and \mathcal{P} , so we have to prove the reverse inclusion. By the first statement, $\mathcal{C}^+ \cap \mathcal{P}$ is

union of simplices in \mathcal{T} . Since $\mathcal{C}^+ \cap \Phi^- = \emptyset$, the vertices of such simplices are in $\Phi^+ \cup \{0\}$, and hence in \mathcal{P}^+ . \square

As a corollary, we obtain the fact that the triangulation \mathcal{T}'_i of Proposition 6.4 inherits the poset structure of $\mathcal{I}_{ab}(\alpha_i)$.

Corollary 6.7. *Let $I \in \mathcal{I}_{ab}(\alpha_i)$ be an abelian ideal. Then the set*

$$\mathcal{T}'_I = \{ \text{Conv}(B(J)) \mid J \in \mathcal{I}_{ab}(\alpha_i), J \subseteq I \}$$

is a triangulation of $\text{Conv}(I)$.

Proof. Let $w \in W^i$ be such that $I = M_i \setminus \overline{N}(w)$ (Proposition 5.4). Then $w(I) \subset \mathcal{P}^+$ and $w(\alpha) \notin \mathcal{P}^+$ for all $\alpha \in M_i \setminus I$. By Theorem 6.6, there exists a subset \mathcal{S}_w of \mathcal{T}'_i such that $w(F_i) \cap \mathcal{P}^+ = \bigcup \{wT \mid T \in \mathcal{S}_w\}$. Since $F_i = \bigcup \{ \text{Conv}(B(J)) \mid J \in \mathcal{I}_{ab}(\alpha_i) \}$, it is clear that $\mathcal{S}_w = \{ \text{Conv}(B(J)) \mid J \in \mathcal{I}_{ab}(\alpha_i), J \subseteq I \}$. The claim follows. \square

Remark 6.1. It is clear that $\mathcal{P}^+ \subseteq \mathcal{P} \cap \mathcal{C}^+$, but in general the equality does not hold. It is immediate that the equality does not hold for Φ of type G_2 . We give a counterexample for Φ of type B_3 . By Theorem 2.3, the simplex generated by $\alpha_2 + 2\alpha_3$, $\alpha_1 + \alpha_2 + 2\alpha_3$, $\alpha_1 + 2\alpha_2 + 2\alpha_3$ is a standard parabolic facet of \mathcal{P} . Its transformed under $s_2s_3s_2$ is the simplex generated by $-\alpha_2$, $\alpha_1 + \alpha_2 + 2\alpha_3$, α_1 , therefore this simplex is a facet. It follows that $\frac{1}{2}(-\alpha_2 + \alpha_1 + \alpha_2 + 2\alpha_3) = \frac{1}{2}(\alpha_1 + 2\alpha_3)$ belongs to the boundary of \mathcal{P} . But the convex linear combinations of α_1 and α_3 belong to the boundary \mathcal{P}^+ , therefore the line through $\frac{1}{2}(\alpha_1 + 2\alpha_3)$ and the origin cuts \mathcal{P}^+ in $\frac{1}{3}(\alpha_1 + 2\alpha_3)$.

7. DIRECTED GRAPHS AND SIMPLICES

The triangulation obtained for \mathcal{P}^+ in the A_n case is the triangulation given by the *anti-standard bases* described in [15]. This can be represented as a special set of trees. We can extend this interpretation to the whole triangulation of \mathcal{P} .

Let us consider the following class of simple directed graphs (no loops, no multiple edges). Given a directed edge e , we write $e = (s, t)$ if s and t are, respectively, the source and the target of e . We call a simple directed graph *n-anti-standard* if it has $[n + 1]$ as vertex set and exactly n directed edges, and satisfies the following properties:

- (1) every vertex is either a source or a target (abelianity);
- (2) for all edges $e = (s, t)$ and $e' = (s', t')$, if $s < s'$ then $t < t'$.

We can make the arcs correspond to the roots in Φ in an obvious way: to the arc (i, j) , we associate the positive root $\alpha_{i,j-1}$, if $i < j$, and the negative root $-\alpha_{j,i-1}$, if $i > j$. So the positive roots are exactly the arcs going from left to

right, if the vertices $1, 2, \dots, n+1$ lie on a horizontal line, from left to right in the usual order. With this correspondence, an n -anti-standard graph corresponds to a subset of Φ of cardinality n . The first of the above conditions says that the corresponding set of roots is *abelian*, in the sense that for any pair of roots in the set, their sum is not a root. The second condition implies that for any two comparable roots in the set, their difference is a root.

The anti-standard directed graphs generalize the concept of anti-standard tree introduced in [15]. There, the authors consider the positive root polytope \mathcal{P}^+ associated with the root system A_n in the usual coordinate description (as settled, for example, in [4]). In the coordinate description, $\text{Span } \Phi$ is the subspace of \mathbb{R}^{n+1} orthogonal to $\sum_{i=1}^{n+1} \varepsilon_i$ and the positive root $\alpha_{i,j}$ is $\varepsilon_i - \varepsilon_{j+1}$, for all $i, j \in [n]$, where $\varepsilon_1, \dots, \varepsilon_{n+1}$ is the standard basis of \mathbb{R}^{n+1} . The edge $e = (h, k)$ corresponds to the root $\varepsilon_h - \varepsilon_k$.

Lemma 7.1. *The n -anti-standard graphs such that $1, \dots, i$ are sources and $i+1, \dots, n+1$ are targets correspond to the simplices of the triangulation \mathcal{T}'_i of the facet F_i .*

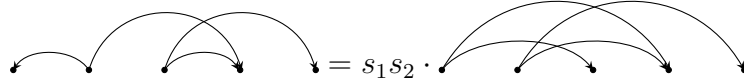
Proof. Clearly, a simplex of the triangulation \mathcal{T}'_i corresponds to an n -anti-standard graphs such that $1, \dots, i$ are sources and $i+1, \dots, n+1$ are targets. Conversely, let G be an n -anti-standard graphs such that $1, \dots, i$ are sources and $i+1, \dots, n+1$ are targets. Let $(h, t_{h_1}), \dots, (h, t_{h_v}), t_{h_1} < t_{h_2} < \dots < t_{h_v}$, be the edges of G with source h , for all $h \in [i]$. By cardinality reasons, $t_{h_v} = t_{(h+1)_1}$, for all $h \in [i-1]$, since otherwise there would not be enough room for n edges. Hence G corresponds to a minimal paths from $\alpha_{1,i}$ to $\alpha_{i,n}$. \square

Theorem 7.2. *The simplices of the triangulation of \mathcal{P} are exactly the sets corresponding to the n -anti-standard graphs.*

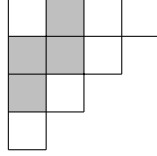
Proof. By Theorem 6.5 and Lemma 7.1, we need to show that the union over all $i \in [n]$ of the orbit of the sets corresponding to n -anti-standard graphs such that $1, \dots, i$ are sources and $i+1, \dots, n+1$ are targets under the action of the quotient W^i gives all the set corresponding to n -anti-standard graphs. Using the coordinate description, W acts as the group of permutations of the standard basis vectors $\varepsilon_1, \dots, \varepsilon_{n+1}$ and this action induces a faithful action on the set of digraphs of vertex set $[n+1]$. It is clear that this action preserves abelianity. The permutations belonging to W^i are exactly the shuffles of the first i nodes with the remaining $n+1-i$ ones, i.e. they are all the permutations σ such that $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(n+1)$. Hence this action preserves also the property of being an n -anti-standard graph.

On the other hand, let G be an n -anti-standard graph and let i be the number of sources of G (so that $n + 1 - i$ is the number of targets by abelianity). Consider the permutation σ which moves all sources in the first i position without changing the relative order among the sources and among the target. Then $\sigma(G)$ corresponds to a minimal paths from $\alpha_{1,i}$ to $\alpha_{i,n}$ and $\sigma^{-1} \in W^i$. We get the assertion. \square

In the next figure we show an anti-standard graph and the distinguished one which it is obtained from.



The graph on the right side of the above picture corresponds to the following minimal path in M_2 , for type A_4 .



The anti-standard graphs whose associated sets of roots are contained in Φ^+ correspond to the anti-standard trees in [15] (see also [25, ex. 6.19-q.]): in fact, the restricted triangulation of Theorem 6.6 is the triangulation of \mathcal{P}^+ given by the *anti-standard bases* studied in [loc. cit.].

We could have done the analogous construction replacing, for all $i \in [n]$, the minimal paths from $\alpha_{1,i}$ to $\alpha_{i,n}$ with the minimal paths from α_i to the highest root θ . The triangulation obtained in this way is the transformed of the previous one by an isometry. We notice that the minimal paths from α_i to θ correspond to the standard bases of [15] that are included in M_i . Looking at the diagram combinatorics, we can easily obtain two bijections between the sets of standard and anti-standard bases. These bijections come, in fact, from isometries of \mathcal{P} that transform the associated triangulations of F_i each into the other. We consider first

$$w_i^r = s_{\alpha_{i_*, i'_*}} \cdots s_{\alpha_{1, i-1}}$$

where $i_* = [i/2]$, $i'_* = [(i+1)/2]$ (integral parts). By a direct check we can see that w_i^r acts on M_i as the antipodal permutation of the columns; therefore w_i^r maps $\alpha_{1,i}$ to α_i , $\alpha_{i,n}$ to θ , and transforms the minimal paths from $\alpha_{1,i}$ to $\alpha_{i,n}$ into the minimal paths from α_i to θ .

The element

$$w_i^c = s_{\alpha_{i, i'_*}} \cdots s_{\alpha_{i+1, n}}$$

where $\bar{i} = i + [(n - i + 1)/2]$, $\bar{i}' = i + [(n - i + 2)/2]$, operates similarly on the rows of M_i , transforming the minimal paths from $\alpha_{1,i}$ to $\alpha_{i,n}$ into the minimal paths from θ to α_i .

8. TRIANGULATION OF \mathcal{P} : TYPE C

Along this section, Φ is a root system of type C_n . Recall that the only long simple root is α_n , the positive short roots are the roots $\alpha_{i,j} = \sum_{k=i}^j \alpha_k$, for all $1 \leq i \leq j \leq n$ except $i = j = n$, and the positive long roots are $\lambda_n = 2(\sum_{k=1}^{n-1} \alpha_k) + \alpha_n$, for $i \in [n]$.

If β is any root not in $M := M_n = \{\alpha \in \Phi^+ \mid c_n(\alpha) = 1\}$, then both $\gamma = \alpha_1 + \cdots + \alpha_n$ and $\gamma' = \alpha_1 + \cdots + \alpha_{n-1}$ are greater than β , and $\gamma + \gamma'$ is a root. Hence all the abelian ideals are contained in M , and M is the unique maximal abelian ideal. By Theorem 2.3, \mathcal{P} has $F := F_n$ as its unique standard parabolic facet; the stabilizer of F is the parabolic subgroup $W\langle\Phi \setminus \alpha_n\rangle$, and the set of all facets of \mathcal{P} is the orbit of F under the action of W .

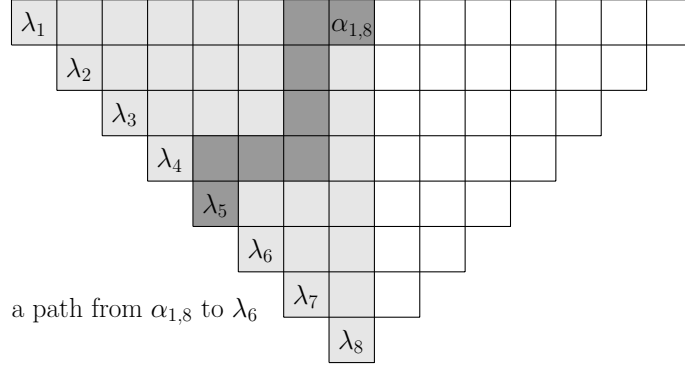
As a matter of fact, the triangulation we shall construct has as vertices not only the vertices of \mathcal{P}_Φ , which are the long roots of Φ , but also the short roots of Φ , which lie in the inner part of edges of \mathcal{P}_Φ (see [5, §5]).

Recall that W^n denotes the set of the minimal left cosets representatives of $W\langle\Phi \setminus \alpha_n\rangle$, which corresponds to the orbit of F under W .

Proposition 8.1. *Let I be an abelian ideal. Then $I \in \mathcal{I}_{ab}(\alpha_n)$ if and only if $\alpha_{1,n} \in I$.*

Proof. Let $\theta = \beta + \gamma$ be a decomposition of θ as a sum of two positive roots. Then exactly one of β and γ belongs to M . Assume $\beta \in M$, so that $\gamma \in \Phi(\Pi \setminus \{\alpha_n\})$. Since $\theta = 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$ and since $\Phi(\Pi \setminus \{\alpha_n\})$ is of type A_{n-1} , we obtain that $\beta \geq \alpha_{1,n}$. Since $\theta = \alpha_{1,n} + \alpha_{1,n-1}$, we obtain that an abelian ideal I satisfies the condition of Proposition 2.2 if and only if $\alpha_{1,n} \in I$. The claim follows. \square

The border strips $B(I)$ of the ideals in $\mathcal{I}_{ab}(\alpha_n)$ are the minimal paths from $\alpha_{1,n}$ to a long root. All such minimal paths consist of $n - 1$ steps, either leftwards, or downwards, starting from $\alpha_{1,n}$, all choices being allowed. In particular, they are 2^{n-1} , in number, and hence $|\mathcal{I}_{ab}(\alpha_n)| = 2^{n-1}$.



Proposition 8.2. *For all abelian ideals $I \in \mathcal{I}_{ab}(\alpha_n)$, $B(I)$ is a basis of the root lattice.*

Proof. Two adjacent roots in minimal path from $\alpha_{1,n}$ to a long root in the root diagram differ by a simple root; the set of the positive differences between adjacent roots in a fixed path is $\{\alpha_1, \dots, \alpha_{n-1}\}$; since $\alpha_{1,n}$ contains α_n with coefficient 1, all the simple roots are integral linear combinations of the roots in the path. \square

This set parametrizes a triangulation of F .

Proposition 8.3. *Let $i \in [n]$. The set*

$$\mathcal{T}'_n = \{\text{Conv}(B(I)) \mid I \in \mathcal{I}_{ab}(\alpha_n)\}$$

is a triangulation F .

Two simplices are adjacent in \mathcal{T}'_n if and only if the corresponding abelian ideals differ for only one element.

In particular, the number of simplices of every root-based triangulation of the standard parabolic facet F equals the cardinality $\mathcal{I}_{ab}(\alpha_n)$.

Proof. We have to prove that $F = \bigcup_{I \in \mathcal{I}_{ab}(\alpha_n)} \text{Conv}(B(I))$ and that, for all $I, I' \in \mathcal{I}_{ab}(\alpha_n)$, $\text{Conv}(B(I)) \cap \text{Conv}(B(I'))$ is a common face of $\text{Conv}(B(I))$ and $\text{Conv}(B(I'))$.

The long positive roots in Φ are $\lambda_i := 2(\sum_{i \leq j \leq n-1} \alpha_j) + \alpha_n$, for all $i \in [n]$, while the short positive roots are $\sum_{i \leq j \leq k} \alpha_j$, for all $i, k \in [n]$. Hence F is the convex hull of the long positive roots. Therefore, in order to prove that $F = \bigcup_{I \in \mathcal{I}_{ab}(\alpha_n)} \text{Conv}(B(I))$, it suffices to prove that any positive linear combination of long positive roots is a positive linear combination of the roots in $B(I)$, for some I in $\mathcal{I}_{ab}(\alpha_n)$.

Let $x = \sum_{i=1}^n x_i \lambda_i$ with $x_i \geq 0$. We will find the appropriate $I \in \mathcal{I}_{ab}(\alpha_n)$ step by step. Since $\lambda_1 + \lambda_n = 2\alpha_{1,n}$, we have:

- (a) if $x_1 > x_n$, then $x = (x_1 - x_n)\lambda_1 + \sum_{i=2}^{n-1} x_i \lambda_i + 2x_n \alpha_{1,n}$;
- (b) if $x_1 < x_n$, then $x = \sum_{i=2}^{n-1} x_i \lambda_i + (x_n - x_1)\lambda_n + 2x_1 \alpha_{1,n}$;

(c) if $x_1 = x_n$, then $x = \sum_{i=2}^{n-1} x_i \lambda_i + 2x_1 \alpha_{1,n} = \sum_{i=2}^{n-1} x_i \lambda_i + 2x_n \alpha_{1,n}$;

hence we may write x as a positive linear combination of either $\{\lambda_1, \dots, \lambda_{n-1}, \alpha_{1,n}\}$, or $\{\lambda_2, \dots, \lambda_n, \alpha_{1,n}\}$. Since, in general, any root $\alpha \in M_n$ is half the sum of the long root in its row and the long root in its column, by iterating this process we get the border of the desired $I \in \mathcal{I}_{ab}(\alpha_n)$. Moreover, generically, the element $x = \sum_{i=1}^n x_i \lambda_i$ belongs to $\text{Conv}(B(I))$ for only one $I \in \mathcal{I}_{ab}(\alpha_n)$: this does not hold only if, at some step, it occurs that we are in case (c). Hence $\text{Vol}(\text{Conv}(B(I)) \cap \text{Conv}(B(I'))) = 0$. By Proposition 8.2, for all $I \in \mathcal{I}_{ab}(\alpha_n)$, $\text{Conv} B(I) \cap \Phi = B(I)$ and therefore $\text{Conv}(B(I)) \cap \text{Conv}(B(I'))$ is a common face of $\text{Conv}(B(I))$ and $\text{Conv}(B(I'))$ for all $I, I' \in \mathcal{I}_{ab}(\alpha_n)$. \square

From the triangulation of F , we can construct a triangulation of the whole \mathcal{P} through the action of the elements in W^n . Hence we obtain the following result.

Theorem 8.4. *Let*

$$\mathcal{T}_n = \{\text{Conv}_0(B(I)) \mid I \in \mathcal{I}_{ab}(\alpha_n)\} \quad \text{and} \quad \mathcal{T} = W^n \mathcal{T}_n.$$

Then \mathcal{T} is a triangulation of \mathcal{P} .

Remark 8.1. The cardinality of W^n is 2^n and the number of border strips in \mathcal{T} is 2^{n-1} . Hence the total number of simplices in \mathcal{T} is 2^{2n-1} .

Actually, this triangulation induces also a triangulation of \mathcal{P}^+ .

To prove this, we need the following lemma. Recall that, for any long positive root λ , we have set

$$\begin{aligned} U_\lambda &:= \{\beta \in \Phi^+ \mid \beta - \lambda \in \Phi^+\}, \\ R_\lambda &:= \{\beta \in \Phi^+ \mid \lambda - \beta \in \Phi^+\} \end{aligned}$$

(see equations (4.1) and the subsequent figure).

Note that U_λ and R_λ both contain only short roots β such that $2\beta - \lambda \in \Phi^+$. Equivalently, given any positive root $\beta \geq \alpha_n$, say $\beta = \sum_{i=i_1}^{i_2} \alpha_i + 2 \sum_{i=i_2+1}^{n-1} \alpha_i + \alpha_n$, the hook centered at β contains the two long positive roots $\lambda_l := 2 \sum_{i=i_1}^{n-1} \alpha_i + \alpha_n$ and $\lambda_d := 2 \sum_{i=i_2+1}^{n-1} \alpha_i + \alpha_n$ satisfying $\lambda_l + \lambda_d = 2\beta$.

Lemma 8.5. *Let $t \in [n]$, $w \in W^n$, and λ be a positive long root.*

- (1) *If $(w(\lambda), \check{\omega}_t) < 0$, then $(w(\beta), \check{\omega}_t) \leq 0$ for all $\beta \in U_\lambda$.*
- (2) *If $(w(\lambda), \check{\omega}_t) > 0$, then $(w(\beta), \check{\omega}_t) \geq 0$ for all $\beta \in R_\lambda$.*

Proof. Let us prove (1). As we have noticed above, $2\beta - \lambda \in \Phi^+$. Since

$$(w(2\beta - \lambda), \check{\omega}_t) = 2(w(\beta), \check{\omega}_t) - (w(\lambda), \check{\omega}_t) \geq 2(w(\beta), \check{\omega}_t) + 1,$$

$(w(\beta), \check{\omega}_t)$ cannot be positive otherwise $(w(2\beta - \lambda), \check{\omega}_t)$ would be ≥ 3 , which is impossible since $m_t \in \{1, 2\}$ for type C_n .

The proof of (2) is analogous. \square

Theorem 8.6. *Let \mathcal{C}^+ denote the positive cone generated by Π . Then $\mathcal{P}^+ = \mathcal{C}^+ \cap \mathcal{P}$ and the triangulation \mathcal{T} of \mathcal{P} restricts to a triangulation of \mathcal{P}^+ .*

Proof. As for the A_n analogue (Theorem 6.6), we shall prove the following two statements, which give the result:

- (1) the triangulation \mathcal{T} restricts to a triangulation of $\mathcal{C}^+ \cap \mathcal{P}$,
- (2) $\mathcal{P}^+ = \mathcal{C}^+ \cap \mathcal{P}$.

The first assertion is equivalent to requiring that every simplex of the triangulation \mathcal{T} is either contained in \mathcal{C}^+ , or intersects it in a null set. Hence, we must show that, for any $w \in W^n$, and $T \in \mathcal{T}_n$, either $w(T) \subseteq \mathcal{C}^+$, or $w(T) \cap \mathcal{C}^+$ has volume equal to 0.

So, let $w \in W^n$, $T \in \mathcal{T}_n$, $T = \text{Conv}_0(B(I))$ with $I \in \mathcal{I}_{ab}(\alpha_n)$, be such that $w(T) \not\subseteq \mathcal{C}^+$. Then, $w(B(I)) \not\subseteq \Phi^+$, hence there exists $\gamma \in B(I)$ and $t \in [n]$ such that $(w(\gamma), \check{\omega}_t) < 0$. We need to prove that, for all $\gamma' \in B(I)$, we have that $(w(\gamma'), \check{\omega}_t) \leq 0$. By Lemma 5.5, we may assume that either

- (1) $\gamma' \geq \gamma$, and γ' and γ lie on the same column in the diagram, or
- (2) $\gamma' \geq \gamma$, and γ' and γ lie on the same row in the diagram.

Suppose we are in case (1). Let λ be the long positive root in the column of γ and γ' . By Lemma 5.5, $(w(\lambda), \check{\omega}_t) < 0$ and we may apply Lemma 8.5, (1). Similarly, if we are in case (2), we may apply Lemma 8.5, (2).

The proof that $\mathcal{P}^+ = \mathcal{P} \cap \mathcal{C}^+$ is similar to that of the A_n analogue in Theorem 6.6. □

As in the A_n case, we obtain, as a corollary of Theorem 8.4, the fact that the triangulation \mathcal{T}'_n of Proposition 8.3 inherits the poset structure of $\mathcal{I}_{ab}(\alpha_n)$.

Corollary 8.7. *Let $I \in \mathcal{I}_{ab}(\alpha_n)$ be an abelian ideal. Then the set*

$$\mathcal{T}'_I = \{ \text{Conv}(B(J)) \mid J \in \mathcal{I}_{ab}(\alpha_n), J \subseteq I \}$$

is a triangulation of $\text{Conv}(I)$.

Note that the triangulation of type C coincides with triangulation studied in [23] on F but not on \mathcal{P}^+ .

9. SOME REMARKS ON VOLUMES

In this section, we show how the curious identity of [11] holds also for the triangulations studied in Sections 6 and 8.

Let us set

$$\text{Vol}(\Pi) := \text{Vol}(\text{Conv}_0(\Pi)).$$

If B is any \mathbb{Z} -basis of the root lattice, then $\text{Vol}(\text{Conv}_0(B)) = \text{Vol}(\Pi)$, since the linear transformation that maps Π to B and its inverse are both integral and hence have determinant 1 or -1 .

By Propositions 6.3 and 8.2, it follows that the total number of simplices in the triangulations of Theorems 6.5 and 8.4 is equal to

$$\frac{\text{Vol}(\mathcal{P})}{\text{Vol}(\Pi)},$$

where $\text{Vol}(\mathcal{P})$ is the volume of \mathcal{P} . Hence, $\frac{\text{Vol}(\mathcal{P}_{A_n})}{\text{Vol}(\Pi_{A_n})} = \binom{2n}{n}$ and $\frac{\text{Vol}(\mathcal{P}_{C_n})}{\text{Vol}(\Pi_{C_n})} = 2^{2n-1}$.

By [14, Theorem 3.11, Lemma 3.12, and Table 4], a triangulation of \mathcal{P}^+ made of simplices generated by integral bases has

$$\prod_{i=1}^n \frac{h + e_i - 1}{e_i + 1}$$

elements, where h is the Coxeter number and the e_i are the exponents of Φ . For Φ of type A_n this number specializes to the n th Catalan number $\frac{1}{n+1}\binom{2n}{n}$, and for type C_n to $\binom{2n-1}{n}$. For type A_n , we could already find this number in [15].

Thus we obtain that $\frac{\text{Vol}(\mathcal{P}_{A_n}^+)}{\text{Vol}(\mathcal{P}_{A_n})} = \frac{1}{n+1}$, and $\frac{\text{Vol}(\mathcal{P}_{C_n}^+)}{\text{Vol}(\mathcal{P}_{C_n})} = \frac{1}{2^{2n-1}}\binom{2n-1}{n} = \frac{1}{2^{2n}}\binom{2n}{n}$. We can easily check that in both cases we have obtained that

$$\frac{\text{Vol}(\mathcal{P}^+)}{\text{Vol}(\mathcal{P})} = \frac{\prod_{i=1}^n e_i}{|W|}.$$

This result is formally similar to the following one, first proved by De Concini and Procesi [11], that holds for all finite crystallographic root systems (see also [13], [2], [28]). Let \mathcal{C}^+ be the positive cone generated by Π and S a sphere centered at the origin. Then

$$\frac{\text{Vol}(\mathcal{C}^+ \cap S)}{\text{Vol}(S)} = \frac{\prod_{i=1}^n e_i}{|W|},$$

where the e_i are the exponents of Φ .

Indeed, we can see that the analogous equality holds for the orbit of each facet of \mathcal{P} , in case A_n and, trivially, in case C_n . Assume that Φ is of type A_n . For any $i \in [n]$, set

$$\tilde{F}_i := \text{Conv}_0(F_i).$$

Consider the union of the orbit of \tilde{F}_i

$$W\tilde{F}_i = \{w(x) \mid w \in W, x \in \tilde{F}_i\}$$

and the sets of simplices

$$W^i\mathcal{T}_i = \{w(S) \mid w \in W^i, S \in \mathcal{T}_i\}.$$

By Theorem 6.6, $W^i\mathcal{T}_i \cap \mathcal{P}^+$ is a triangulation of $W\tilde{F}_i \cap \mathcal{P}^+$. We have the following result.

Proposition 9.1. *For each $i \in [n]$,*

$$\frac{|W^i \mathcal{T}_i \cap \mathcal{P}^+|}{|W^i \mathcal{T}_i|} = \frac{1}{n+1}.$$

Proof. For any basis Π' of Φ , let $\mathcal{C}_{\Pi'}^+$ denote the positive cone generated by Π' . Since W is a group of isometries and is transitive on the set of bases of Φ , for any fixed standard parabolic facet F_i we have that

$$\text{Vol}(W\tilde{F}_i \cap \mathcal{C}_{\Pi'}^+) = \text{Vol}(W\tilde{F}_i \cap \mathcal{C}^+) = \text{Vol}(W\tilde{F}_i \cap \mathcal{P}^+)$$

for all bases Π' of Φ .

It is easy to see that, for any finite crystallographic irreducible Φ , $\text{Span } \Phi$ is the union of the $n+1$ positive cones generated by the sets Π and $\{-\theta\} \cup \Pi \setminus \{\alpha_k\}$, for all $k \in [n]$, and that, moreover, these cones have pairwise null intersections (a proof can be found in [11]).

If Φ is of type A_n , the set $\{-\theta\} \cup \Pi \setminus \{\alpha_k\}$ is a basis of Φ , for all $k \in [n]$, and hence the volume of the intersection of $W\tilde{F}_i$ with each of these cones is the same and is equal to $\frac{1}{n+1} \text{Vol}(W\tilde{F}_i \cap \mathcal{P})$, so that

$$\text{Vol}(W\tilde{F}_i \cap \mathcal{P}^+) = \frac{1}{n+1} \text{Vol}(W\tilde{F}_i \cap \mathcal{P}).$$

Since all the simplices in \mathcal{T} have the same volume, and both $W\tilde{F}_i \cap \mathcal{P}^+$ and $W\tilde{F}_i \cap \mathcal{P}$ are union of simplices in \mathcal{T} , this proves the claim. \square

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